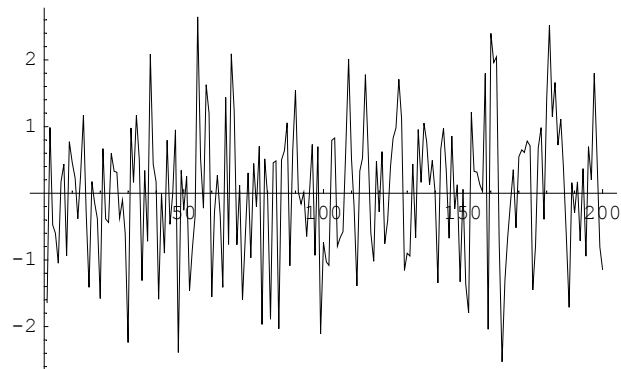


Vremenski nizovi

Elementarni primjeri

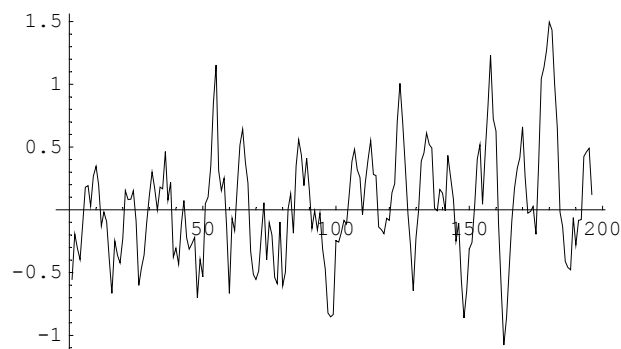
■ Bijeli šum

Bijeli šum je niz nezavisnih jednako distribuiranih slučajnih varijabli. Simulacija bijelog šuma iz normalne $N(0, \sigma^2)$ distribucije.



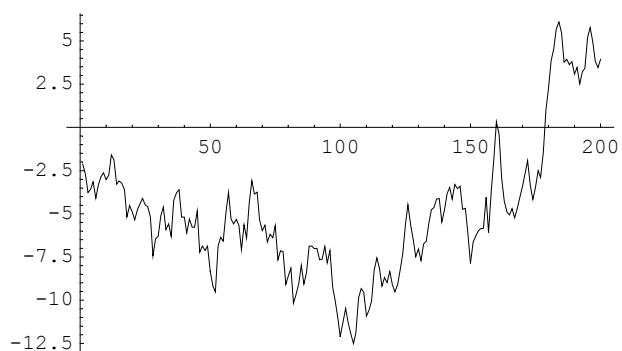
■ Proces pokretnih sredina

Proces pokretnih sredina uz Y_j normalne $N(0, \sigma^2)$ slučajne varijable



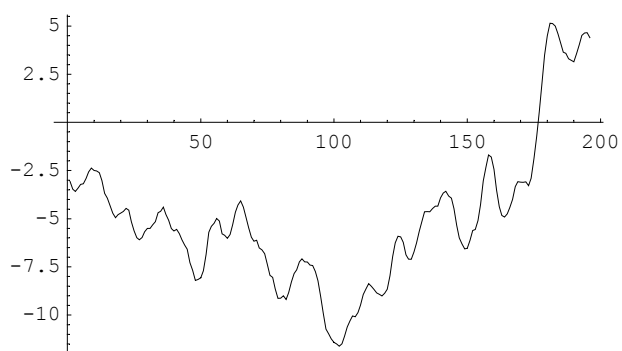
■ Slučajna šetnja

Slučajna šetnja s koracima iz normalne $N(0, \sigma^2)$ distribucije.



■ Izgladivanje slučajne šetnje

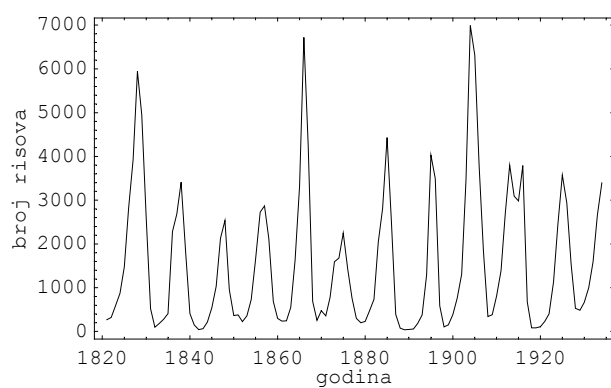
Proces pokretnih sredina uz Y_j slučajna šetnja s normalnim koracima.



Primjeri vremenskih nizova

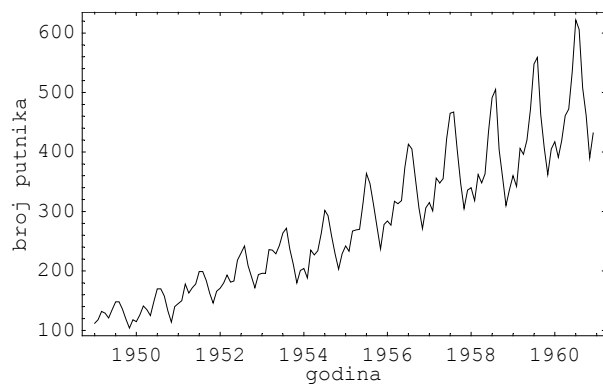
■ Primjer 1:

Godišnji broj uhvaćenih risova u sjeverozapadnoj Kanadi od 1821 do 1934.

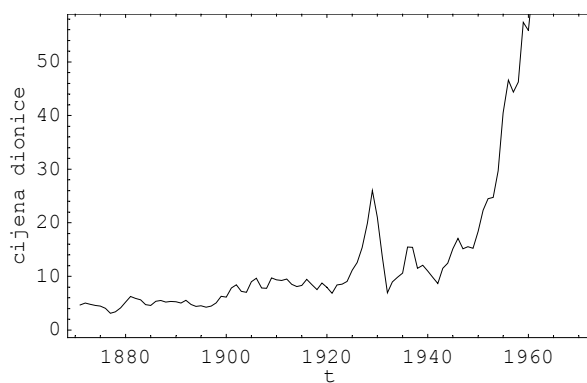


■ Primjer 2:

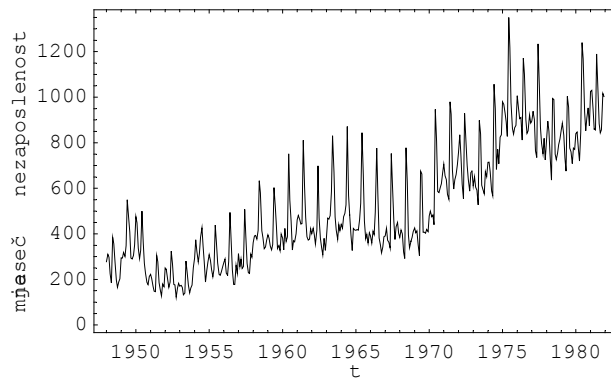
Ukupni mjesečni broj internacionalnih avionskih putnika u tisućama od siječnja 1949 do prosinca 1960.

**■ Primjer 3:**

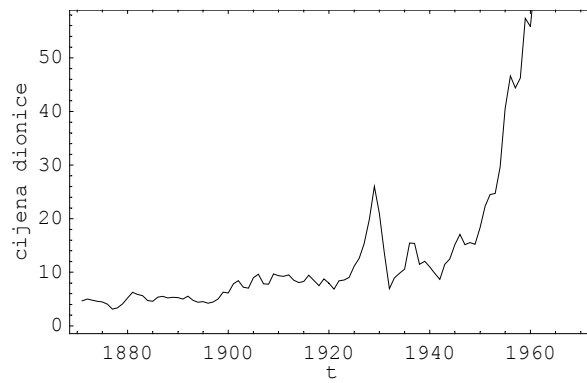
Cijena dionice od 1871 do 1970

**■ Primjer 4:**

Mjesečna nezaposlenost (u tisućama) za muške u dobi 16-19 godina od 1948-1981.

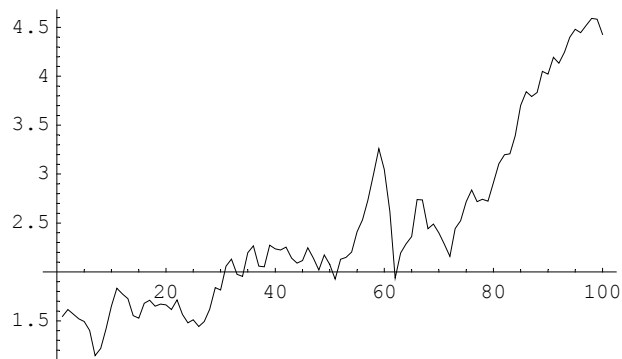


Procjena i eliminacija trenda



■ Eliminiranje varijabilnosti

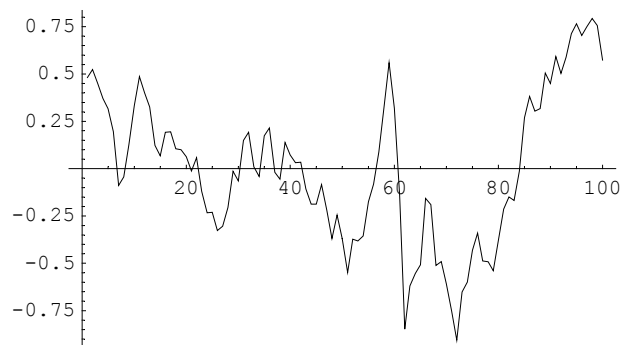
Box-Coxova logaritamska transformacija podataka.



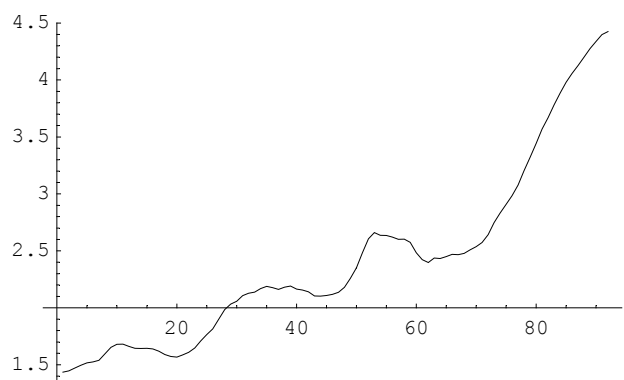
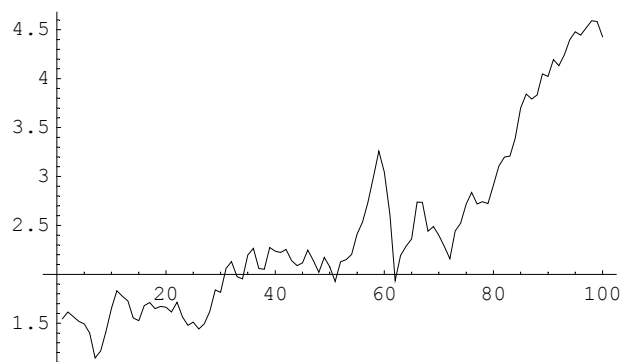
■ Prilagođavanje polinoma

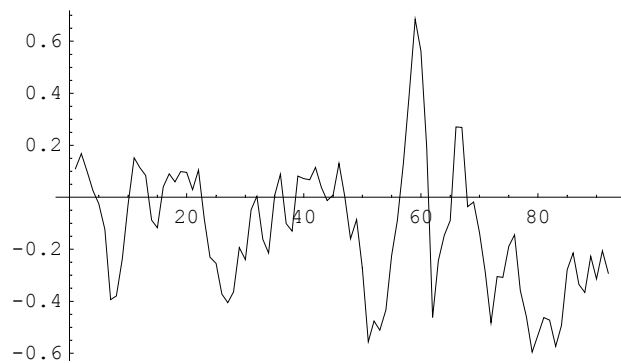
Podaci sugeriraju linearan trend, pa prilagođujemo linearnu funkciju.

$$1.03569 + 0.0281928 t$$

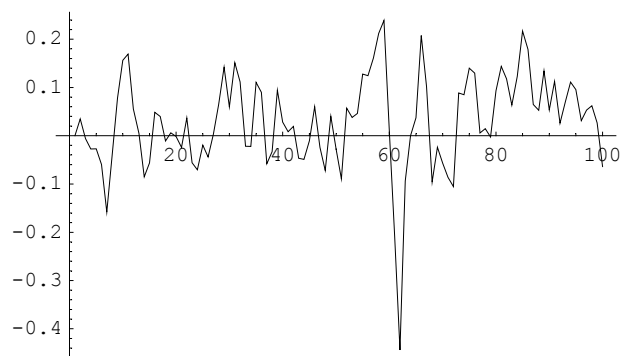
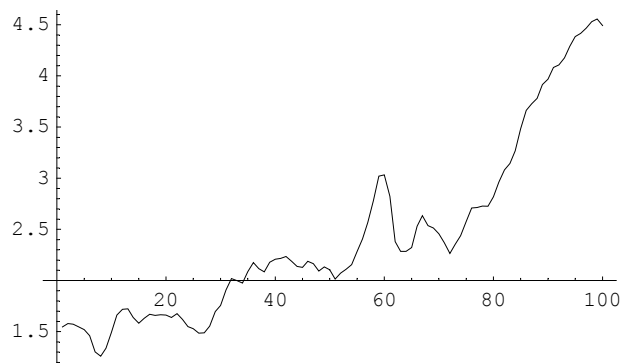
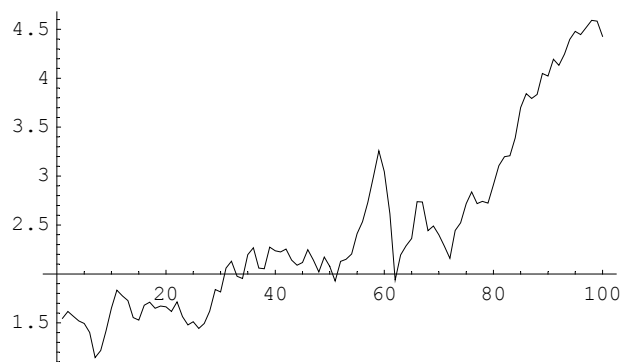


■ Izgladivanje filterom pokretnih sredina

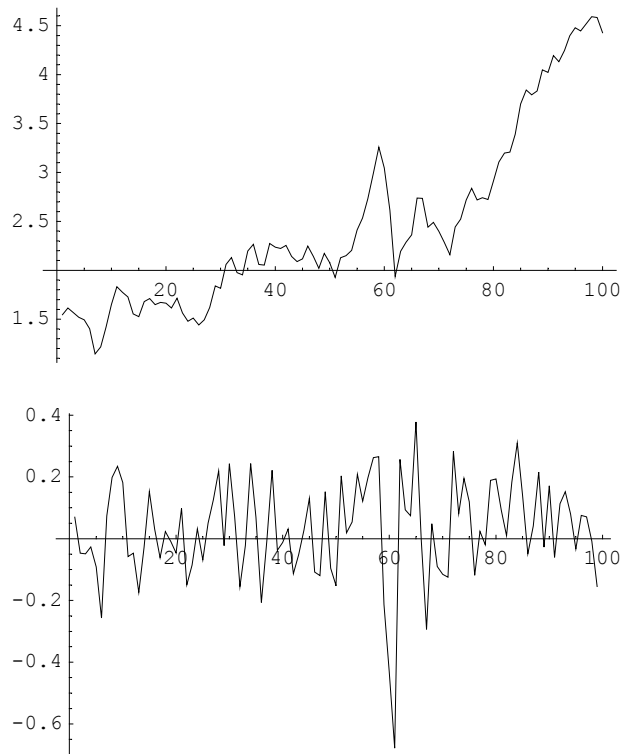




■ Eksponencialno izgladivanje

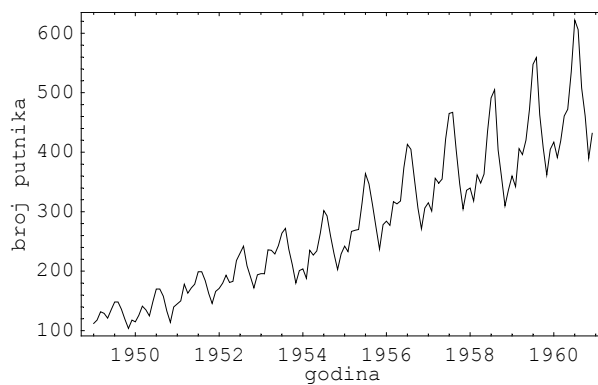


■ Diferenciranje



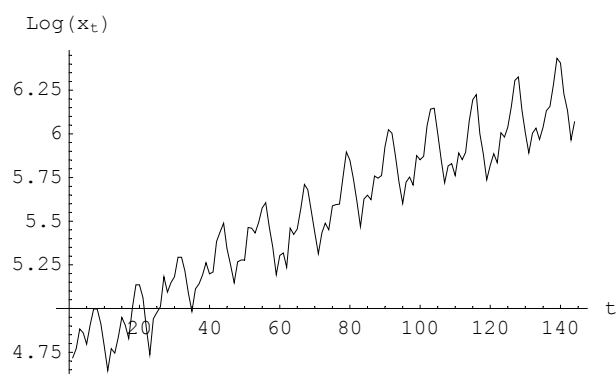
Procjena i eliminacija sezonske komponente

Ukupni mjesečni broj internacionalnih avionskih putnika u tisućama od siječnja 1949 do prosinca 1960.



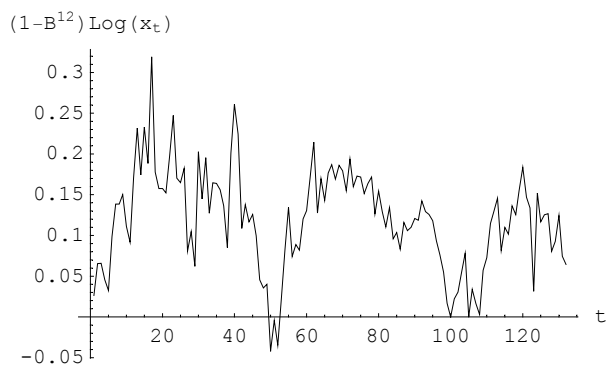
■ Eliminiranje varijabilnosti

Box-Coxova logaritamska transformacija podataka.



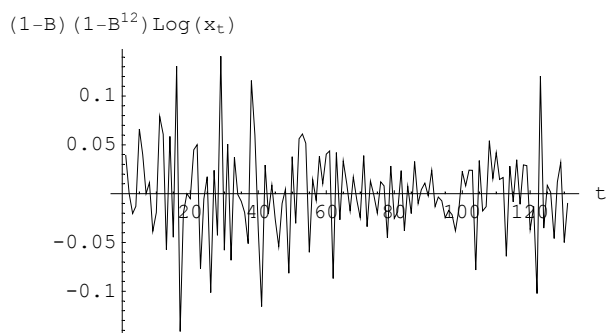
■ Eliminiranje sezonske komponente diferenciranjem

Diferenciramo s periodom 12.



Podaci više ne pokazuju periodičnost, t.j., sezonska komponenta je eliminirana.

Diferenciramo još jedanput. Dobivamo:



Transformirani podaci izgledaju kao vjerojatna realizacija stacionarnog niza.

ARMA modeli

Opći oblik ARMA modela

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \dots + \theta_q Z_{t-q}.$$

Konstante $\{\phi_1, \phi_2, \dots, \phi_p\}$ i $\{\theta_1, \theta_2, \dots, \theta_q\}$ zovu se *autoregresivni (AR) koeficijenti* i *koeficijenti pokretnih sredina (MA)*. ($\{Z_t\}$ je bijeli šum s očekivanjem 0 i varijancom σ^2 .)

Uvedimo *operator pomaka unatrag B* pomoću

$$B^j X_t = X_{t-j}.$$

Definiramo *autoregresivni polinom $\phi(x)$* kao

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \dots - \phi_p x^p$$

i *polinom pomičnih sredina $\theta(x)$* kao

$$\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \dots + \theta_q x^q,$$

te pretpostavimo da ti polinomi nemaju zajedničkih nul-točaka. Tada se definicijska jednadžba ARMA procesa može zapisati kao

$$\phi(B) X_t = \theta(B) Z_t.$$

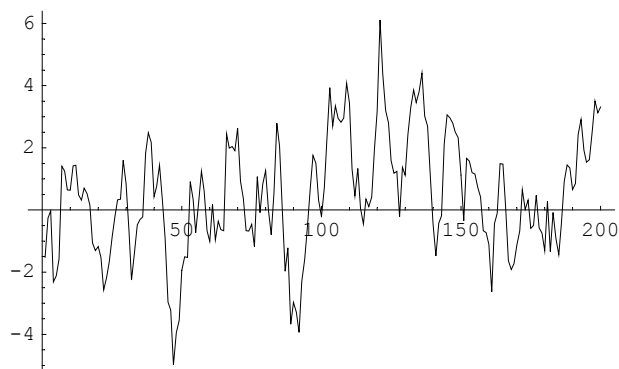
■ Autoregresivni procesi AR(p)

Kod autoregresivnih procesa $q=0$. Autoregresivan proces je stacionaran ako su nul-točke autoregresivnog polinoma izvan jedinične kružnice.

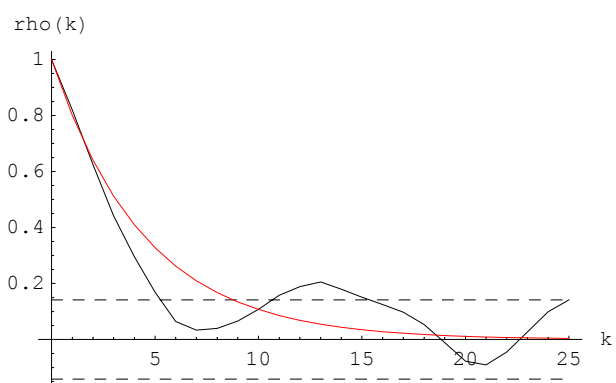
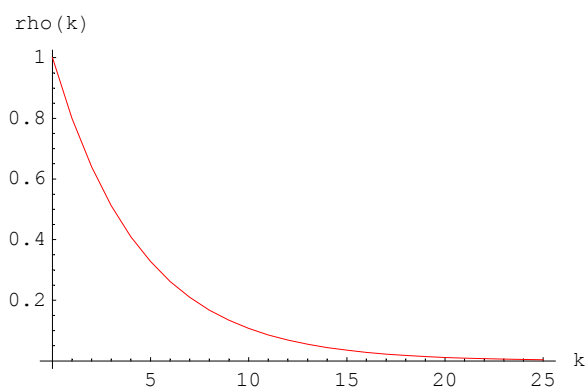
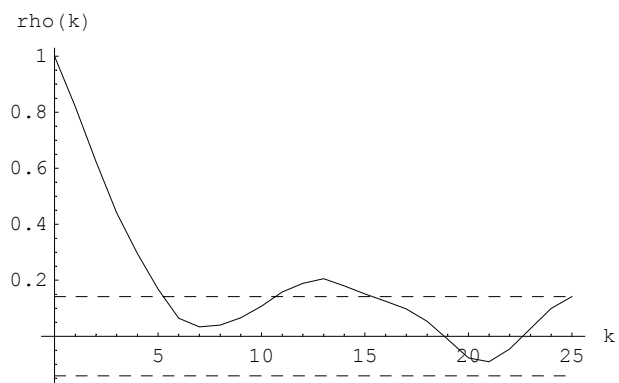
Primjeri:

AR(1) procesi:

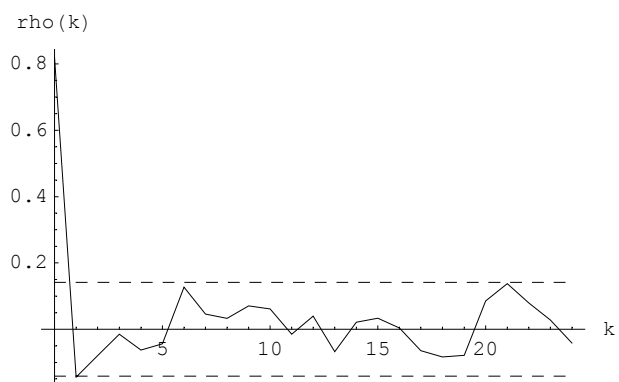
$$X_t = 0.8 X_{t-1} + Z_t, Z_t \text{ je WN}(0,1)$$

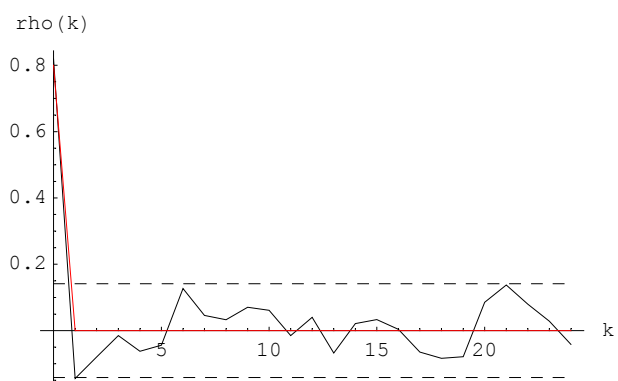
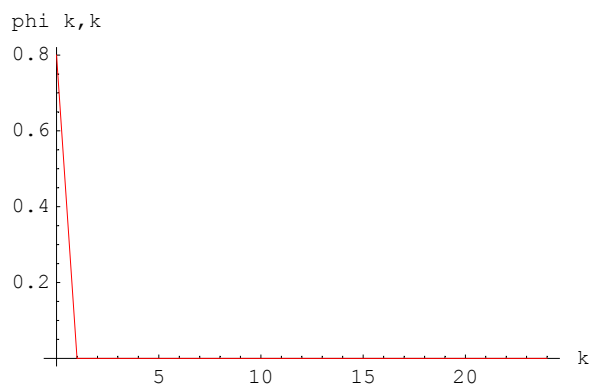


Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.



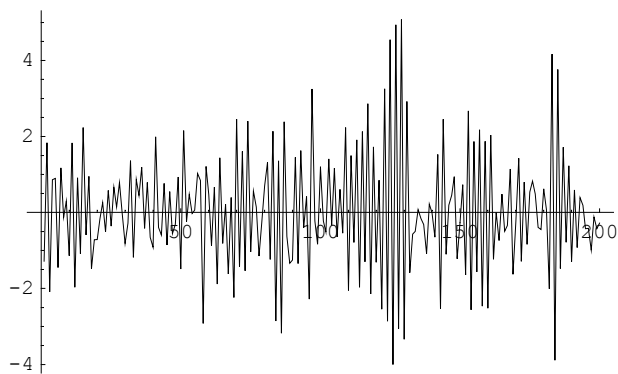
Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.



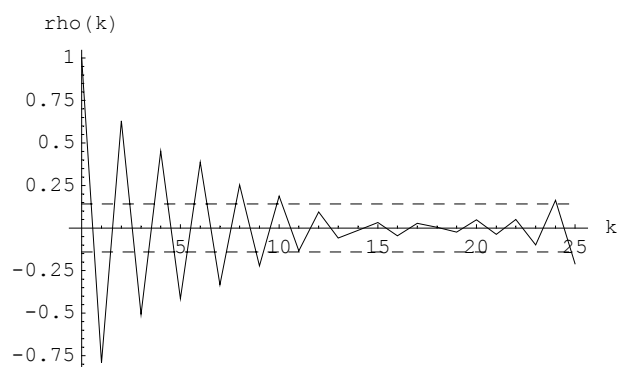


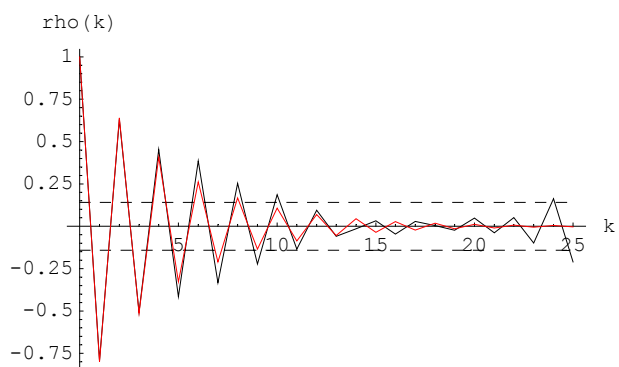
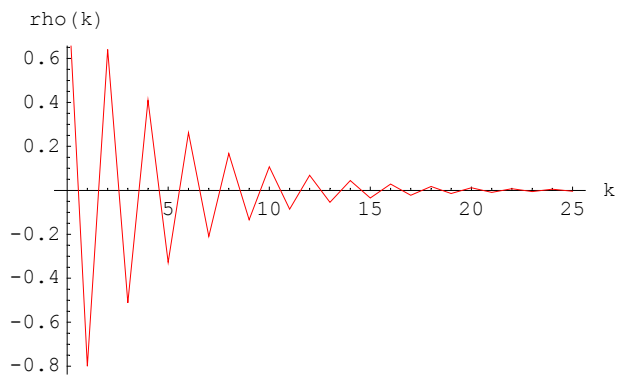
AR(1) procesi:

$$X_t = -0.8 X_{t-1} + Z_t, Z_t \text{ je WN}(0,1)$$

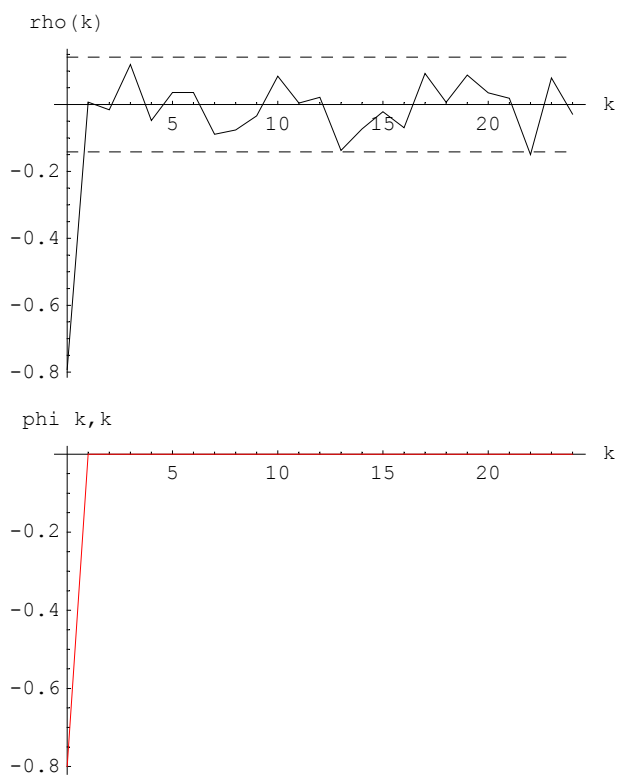


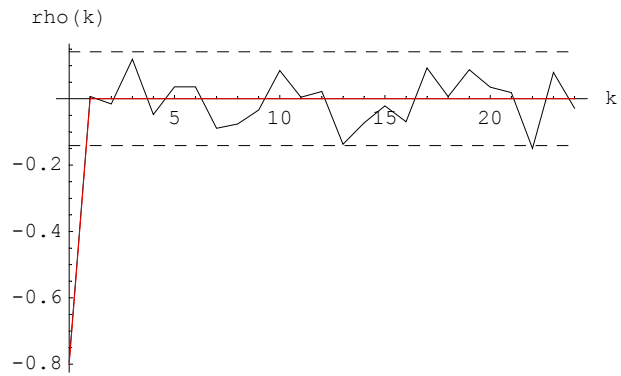
Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.





Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.



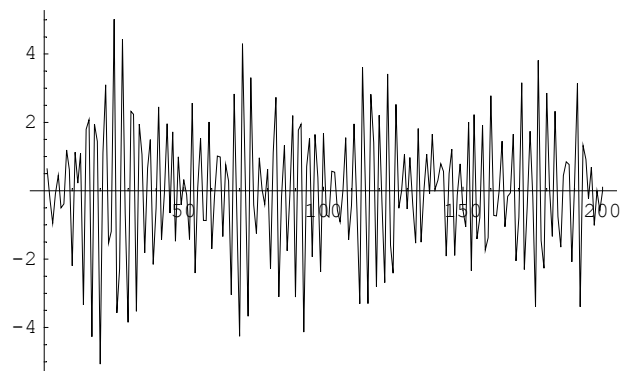


AR(2) procesi:

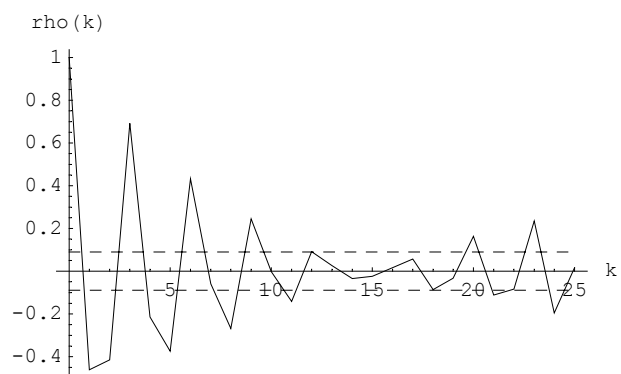
$$X_t = -0.9 X_{t-1} - 0.8 X_{t-2} + Z_t, Z_t \text{ je } WN(0,1)$$

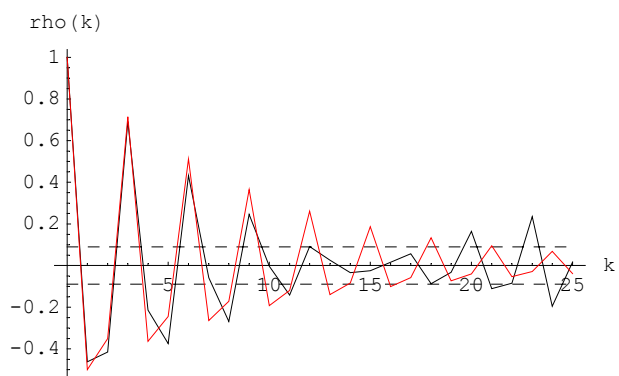
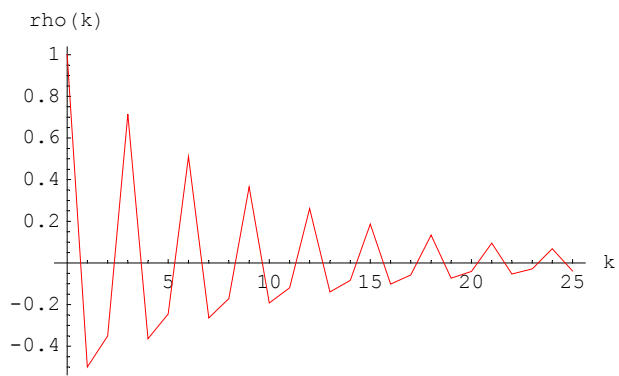
Nul-točke autoregresivnog polinoma su

$$\{x \rightarrow -0.5625 - 0.966227 i\}, \{x \rightarrow -0.5625 + 0.966227 i\}$$

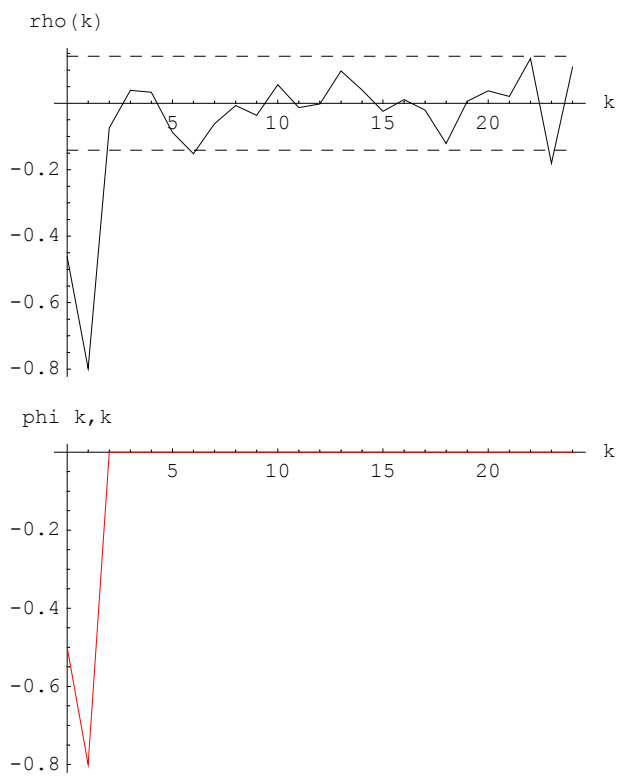


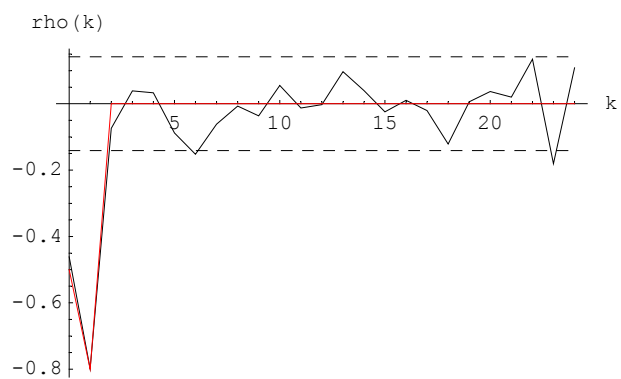
Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.





Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.

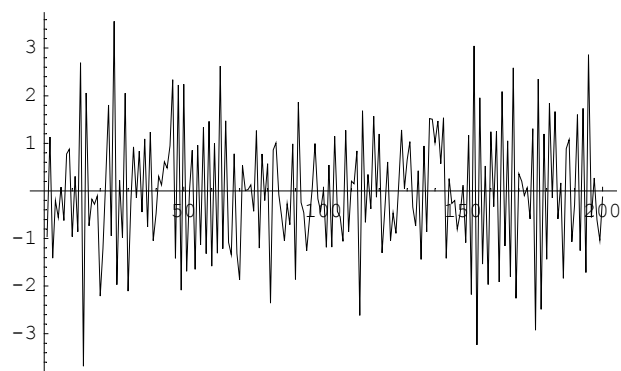




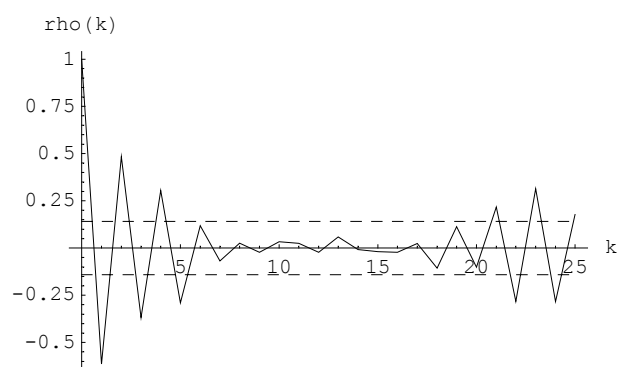
AR(2) procesi:

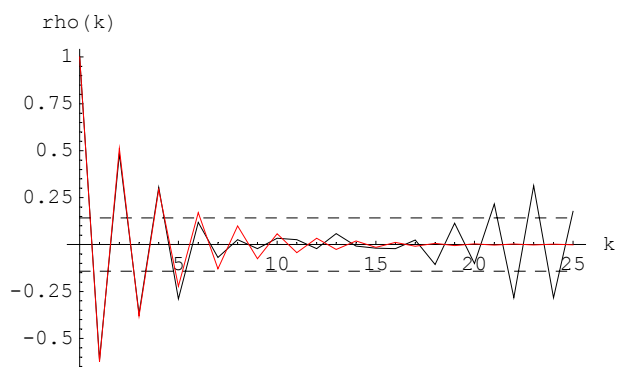
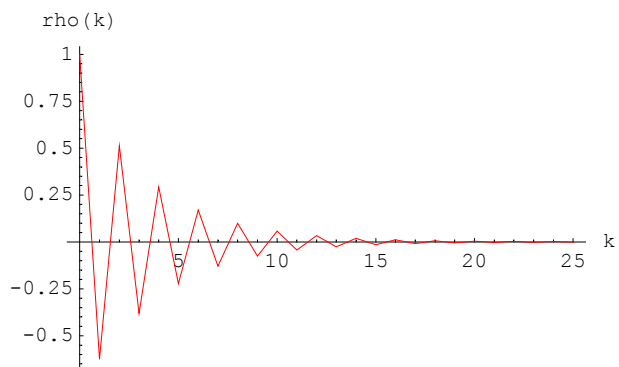
$$X_t = -0.5 X_{t-1} + 0.2 X_{t-2} + Z_t, \quad Z_t \text{ je } \text{WN}(0,1)$$

Nul-točke autoregresivnog polinoma su $\{x \rightarrow -1.31174\}, \{x \rightarrow 3.81174\}$

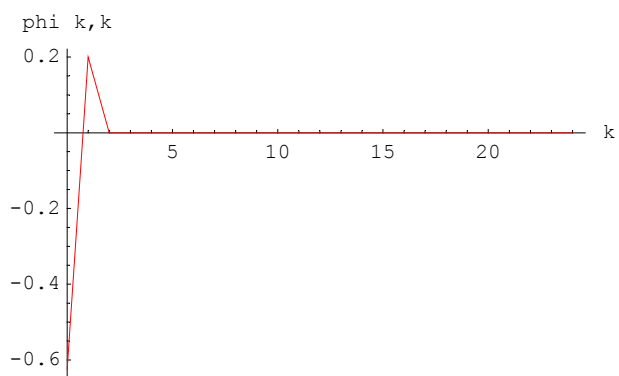
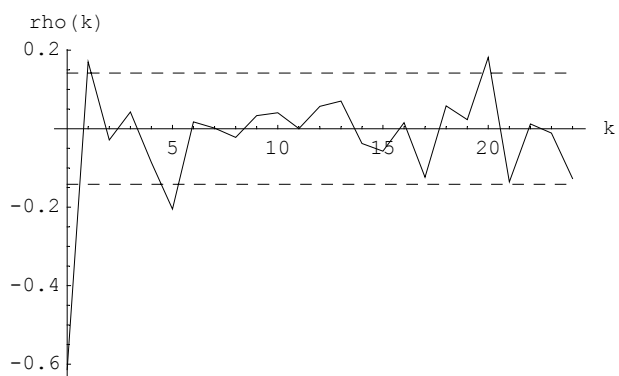


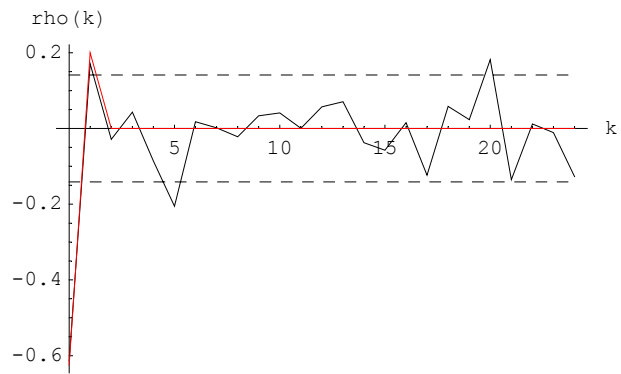
Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.





Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.



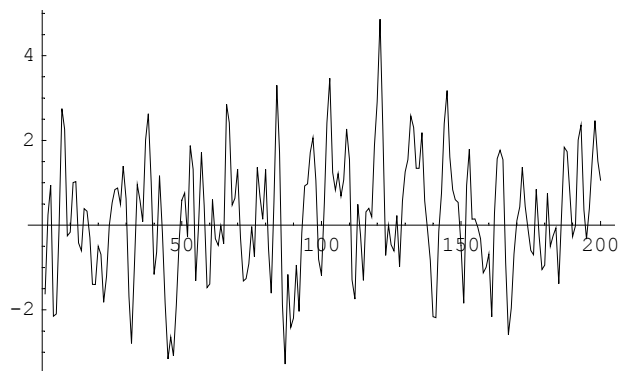


■ Procesi pokretnih sredina MA(q)

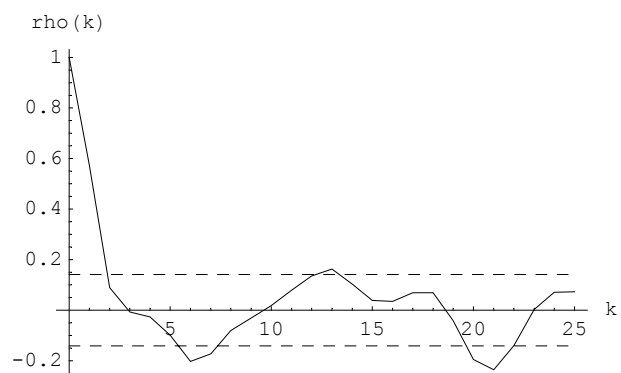
Kod procesa pokretnih sredina $p=0$.

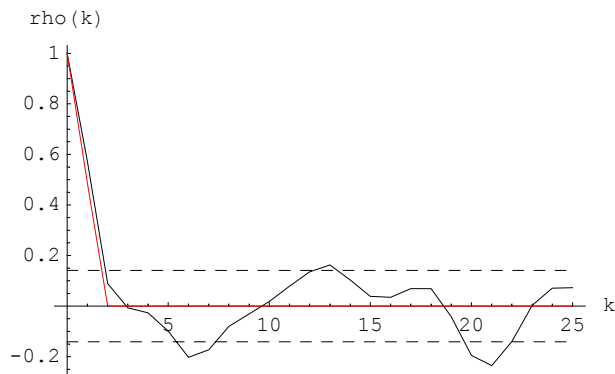
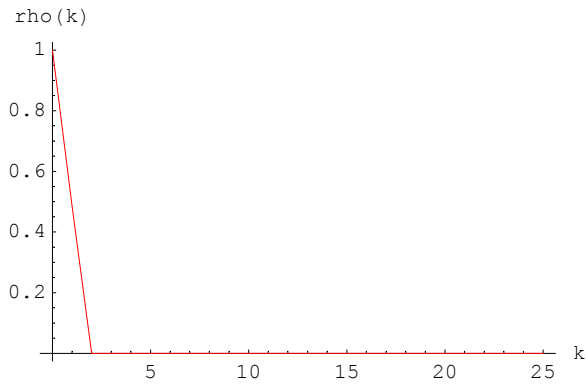
MA(1) procesi:

$$X_t = Z_t + 0.8 Z_{t-1}, \quad Z_t \text{ je WN}(0,1)$$

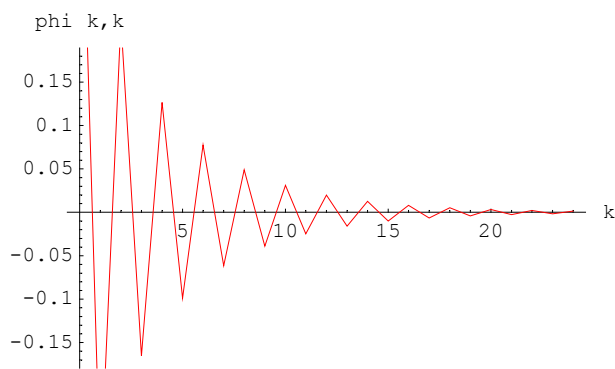
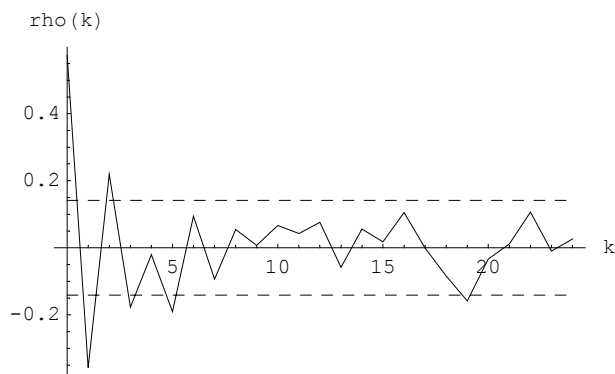


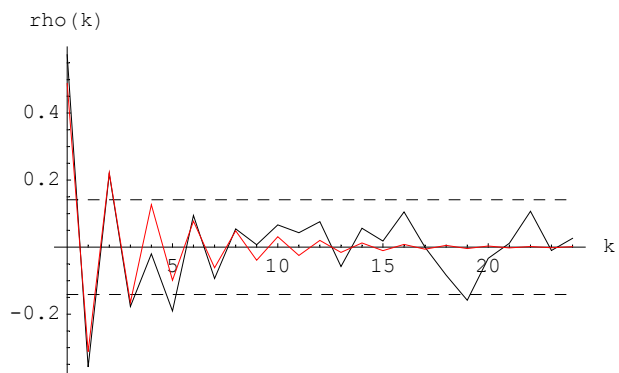
Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.





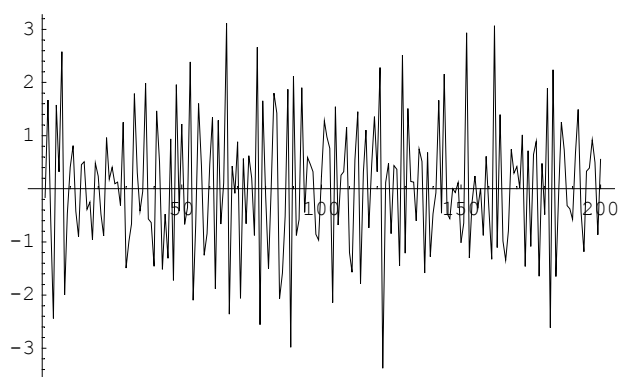
Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.



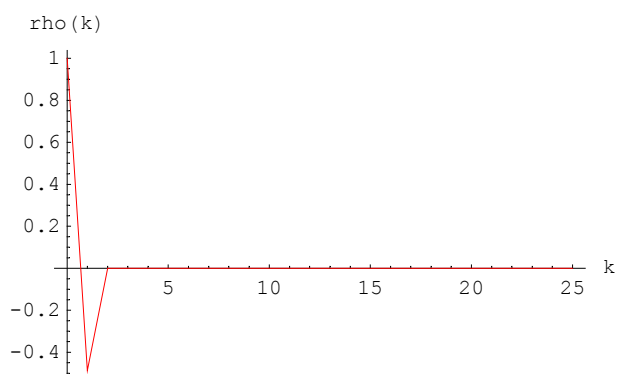
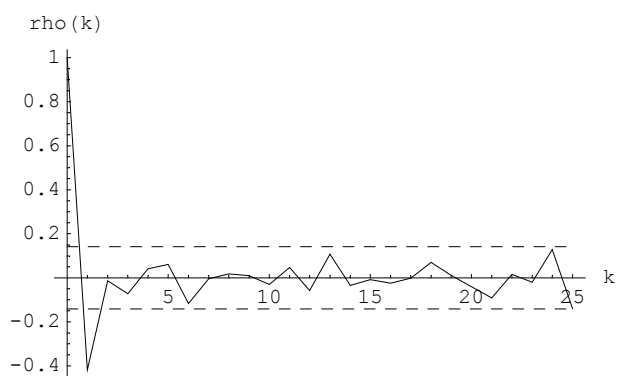


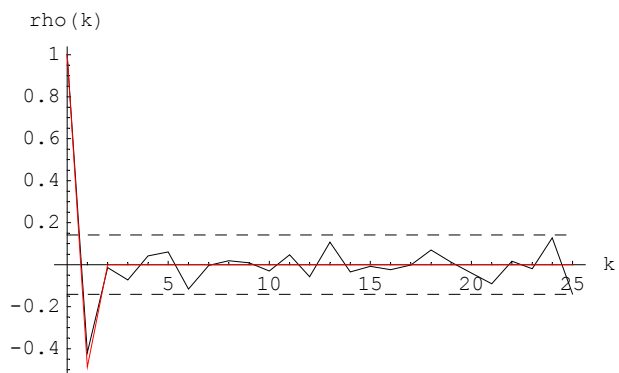
MA(1) procesi:

$$X_t = Z_t - 0.8 Z_{t-1}, \quad Z_t \text{ je WN}(0,1)$$

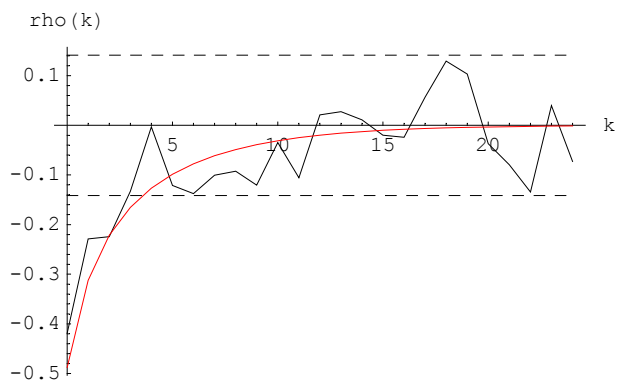
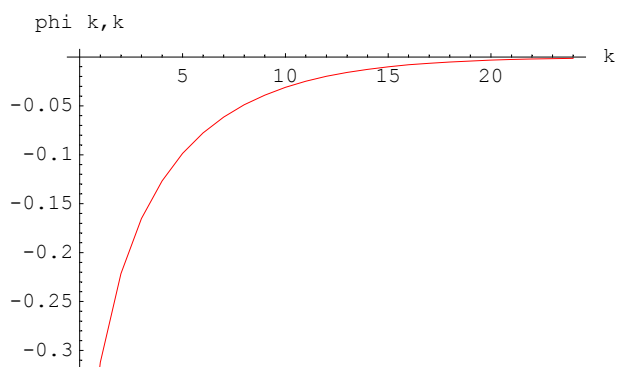
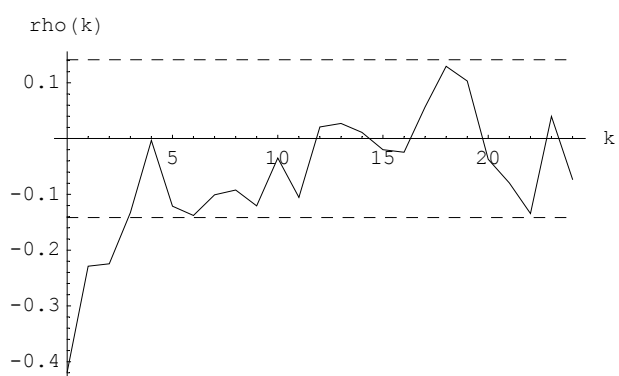


Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.





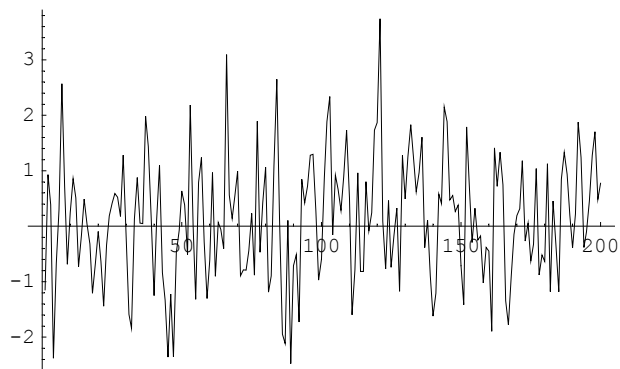
Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.



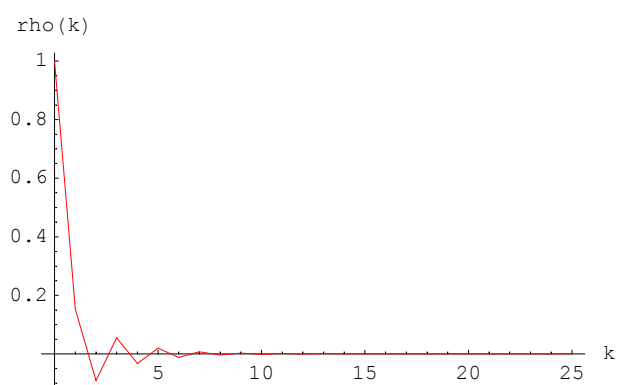
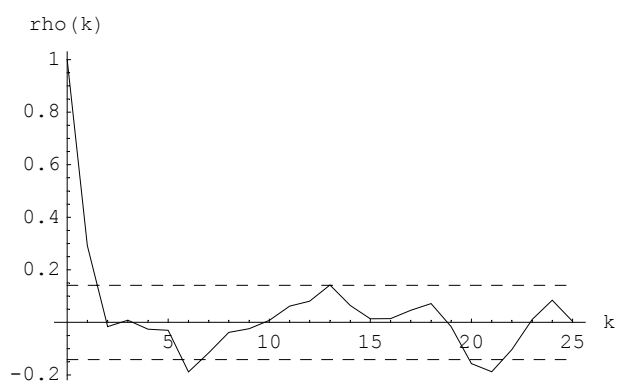
■ ARMA procesi ARMA(p,q)

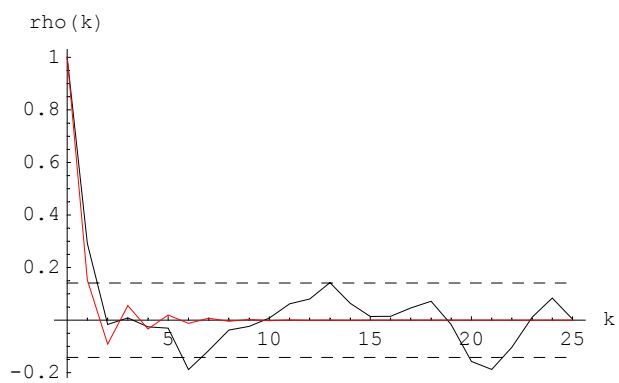
ARMA(1,1) procesi:

$$X_t = 0.6 X_{t-1} + Z_t + 0.8 Z_{t-1}, \quad Z_t \text{ je WN}(0,1)$$

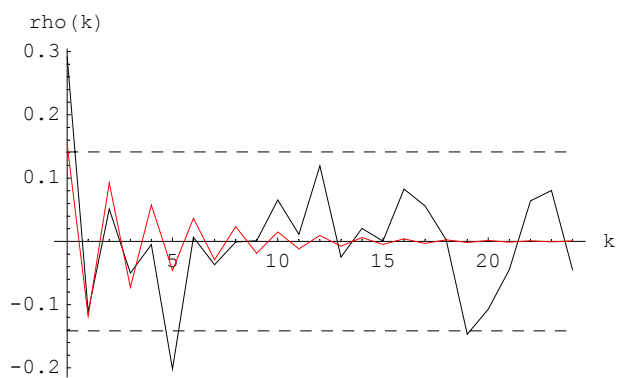
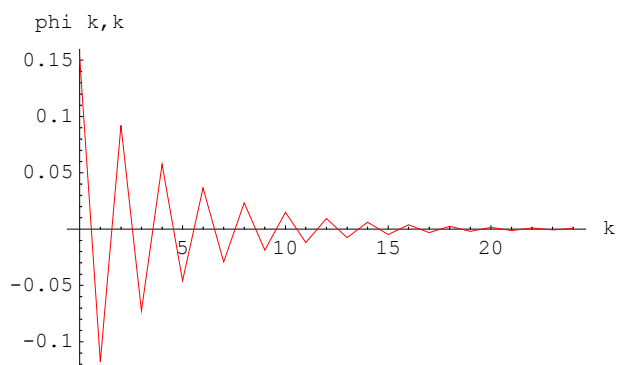
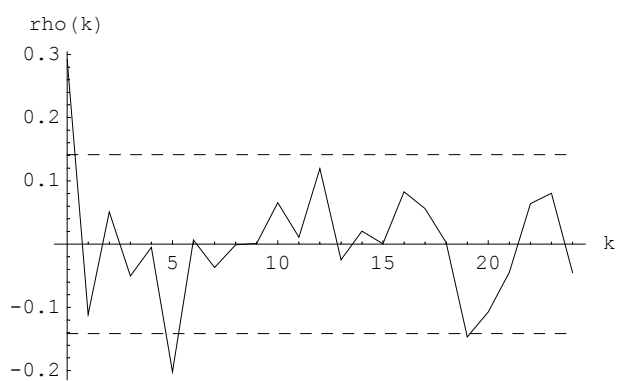


Autokorelacijska funkcija uzorka i teorijska autokorelacijska funkcija.





Parcijalna korelacijska funkcija uzorka i teorijska parcijalna korelacijska funkcija.



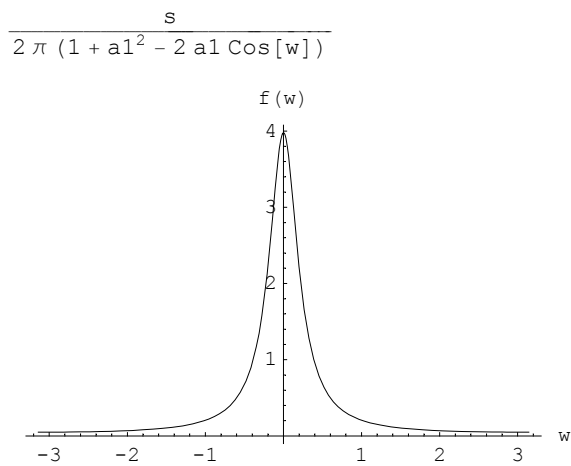
Analiza u frekvencijskoj domeni

Primjeri spektralne gustoće ARMA procesa.

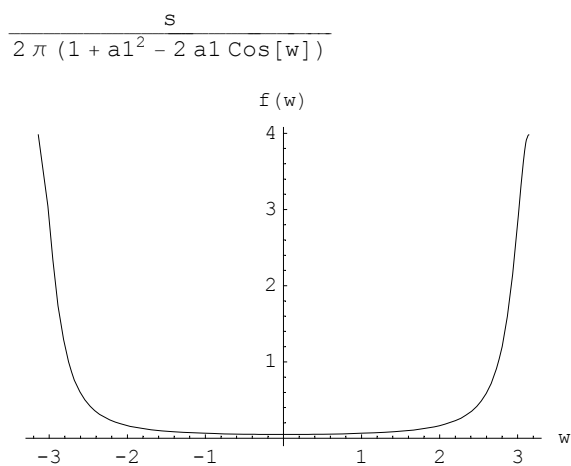
AR(1) procesi:

$$X_t = a_1 X_{t-1} + Z_t, Z_t \text{ je WN}(0,1)$$

$$a_1 = 0.8$$



$$a_1 = -0.8$$

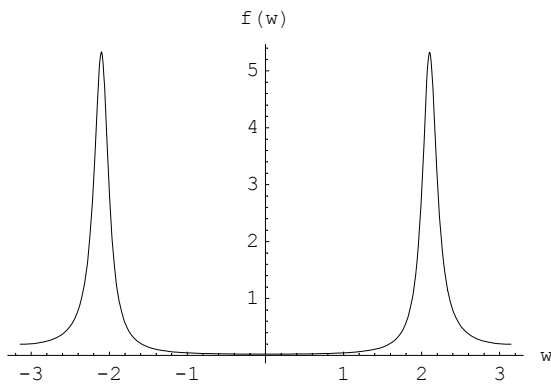


AR(2) procesi:

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + Z_t, Z_t \text{ je WN}(0,1)$$

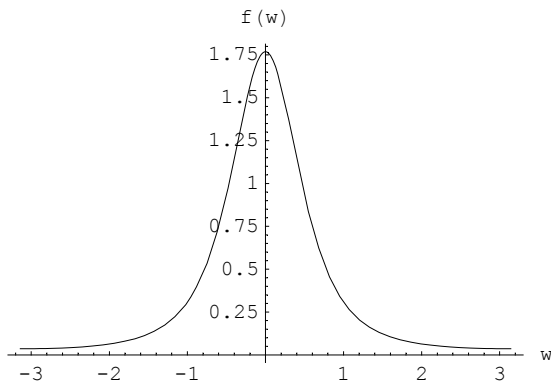
$$a_1 = -0.9, a_2 = -0.8$$

$$s / (2 \pi (1 + a_1^2 + a_2^2 + 2 a_1 (-1 + a_2) \cos [w] - 2 a_2 \cos [2 w]))$$



$$a_1 = 0.9, \quad a_2 = -0.2$$

$$s / (2 \pi (1 + a_1^2 + a_2^2 + 2 a_1 (-1 + a_2) \cos [w] - 2 a_2 \cos [2 w]))$$

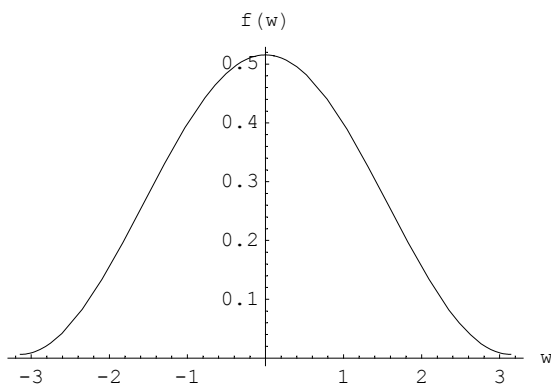


MA(1) procesi:

$$X_t = Z_t + b_1 Z_{t-1}, \quad Z_t \text{ je WN}(0,1)$$

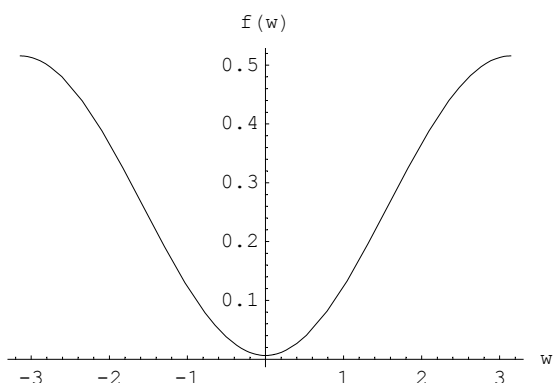
$$b_1 = 0.8$$

$$\frac{s (1 + b_1^2 + 2 b_1 \cos [w])}{2 \pi}$$



$$b_1 = -0.8$$

$$\frac{s (1 + b_1^2 + 2 b_1 \cos[w])}{2 \pi}$$

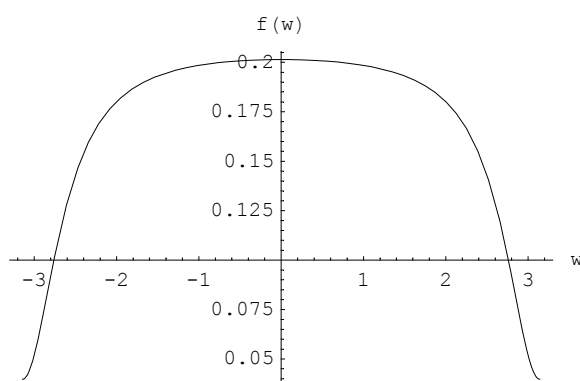


ARMA(1,1) procesi:

$$X_t = a_1 X_{t-1} + Z_t + b_1 Z_{t-1}, Z_t \text{ je } WN(0,1)$$

$$a_1 = -0.6, b_1 = 0.8$$

$$\frac{s (1 + b_1^2 + 2 b_1 \cos[w])}{2 \pi (1 + a_1^2 - 2 a_1 \cos[w])}$$

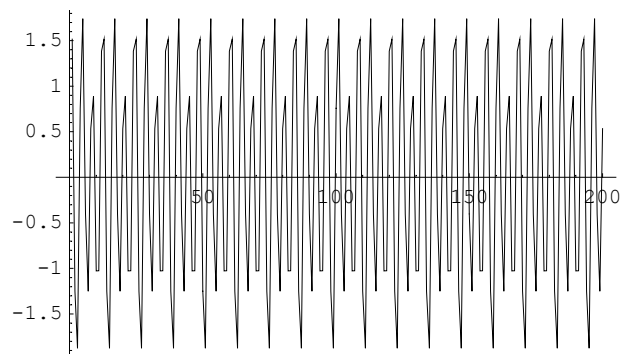


Primjer procesa s diskretnim spektrom:

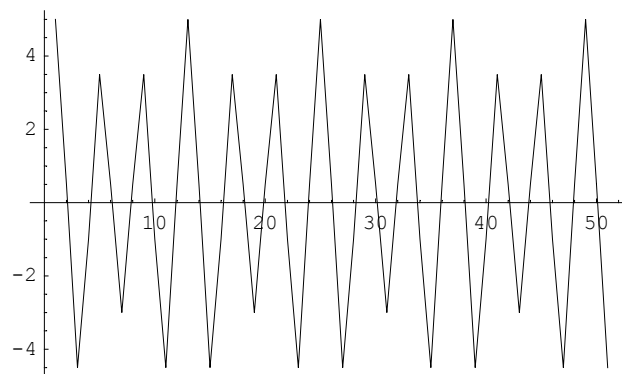
$$X_n = \sum_{j=0}^1 [A_j \cos(\omega_j n) + B_j \sin(\omega_j n)]$$

A_0, B_0 su $N(0,1)$ slučajne varijable, A_1, B_1 su $N(0,2)$ slučajne varijable,

$$\omega_0 = \frac{\pi}{3}, \omega_1 = \frac{\pi}{2}.$$



Autokorelacijska funkcija:

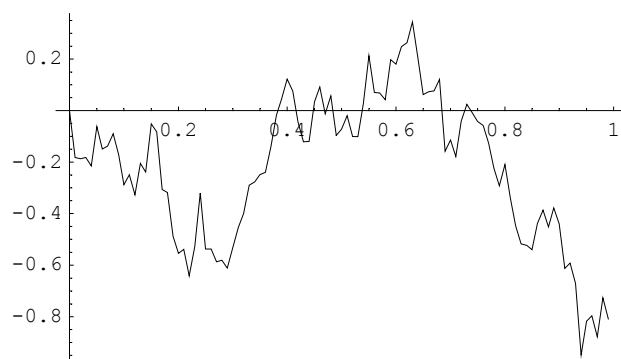


Brownovo gibanje i difuzije

Standardno Brownovo gibanje

Standardno Brownovo gibanje je stohastički proces $(B_t, t > 0)$ s nezavisnim stacionarnim prirastima, takav da B_1 ima $N(0, 1)$ distribuciju.

■ Simulacija standardnog Brownovog gibanja



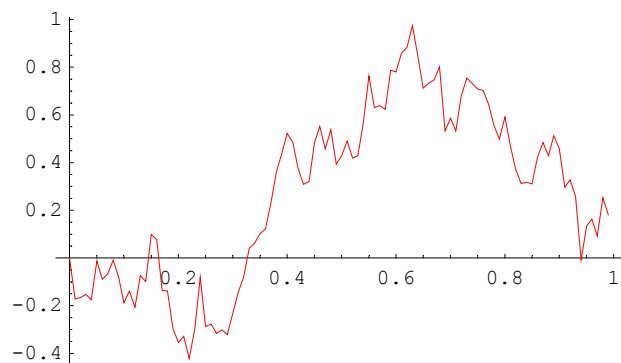
Brownovo gibanje

Brownovo gibanje s koeficijentom drifta μ i difuzijskim koeficijentom σ je stohastički proces $(W_t, t > 0)$ s nezavisnim i stacionanim prirastima takav da W_1 ima $N(\mu, \sigma^2)$ distribuciju.

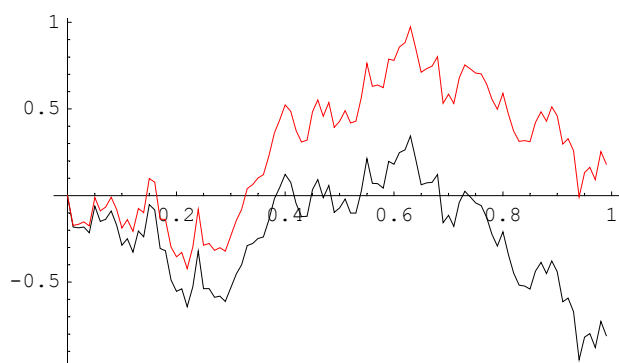
Brownovo gibanje W može se dobiti iz standardnog Brownovog gibanja B linearnom transformacijom $W_t = \mu t + \sigma B_t$.

■ Simulacija Brownovog gibanja

$$dW_t = 1 dt + 1 dB_t \quad ; \quad W_0 = 0$$

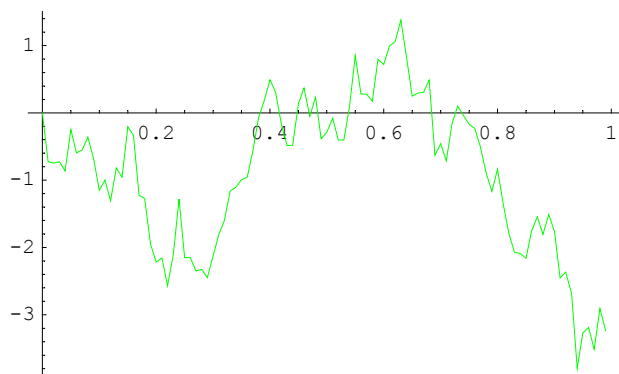


Usporedba Brownovog gibanja bez drifta i s driftom.

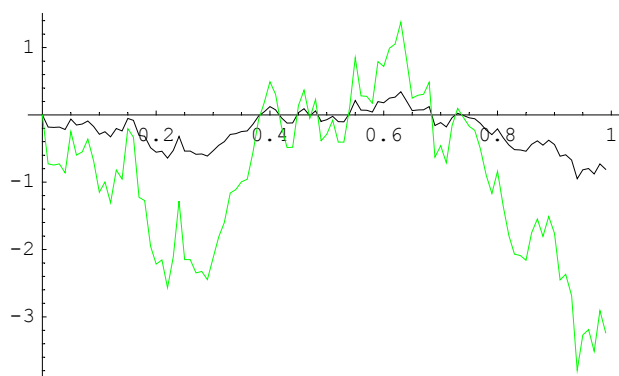


Utjecaj koeficijenta difuzije σ na trajektoriju.

$$dW_{2t} = 0 dt + 4 dB_t \quad ; \quad W_{2_0} = 0$$



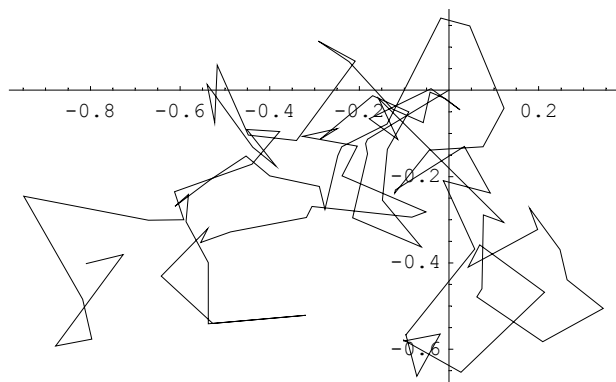
Usporedba Brownovih gibanja za $\sigma=1$ i $\sigma=4$.



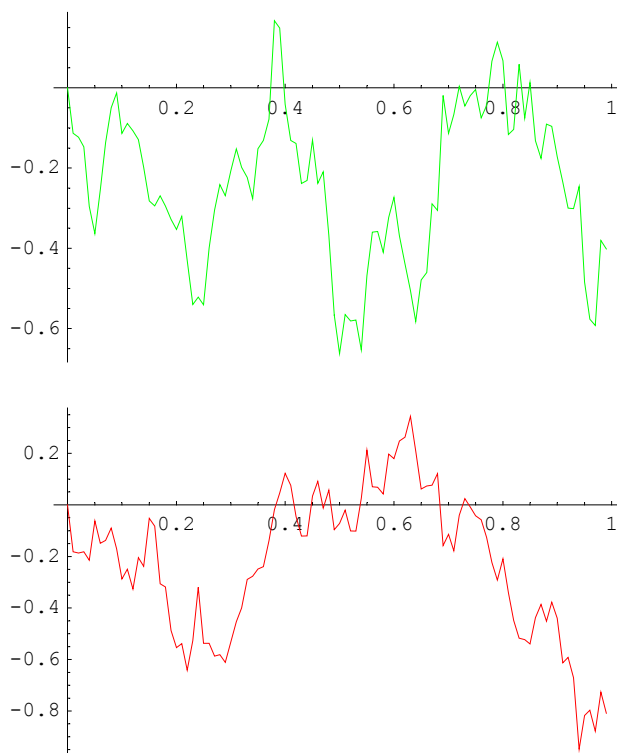
Dvodimenzionalno Brownovo gibanje

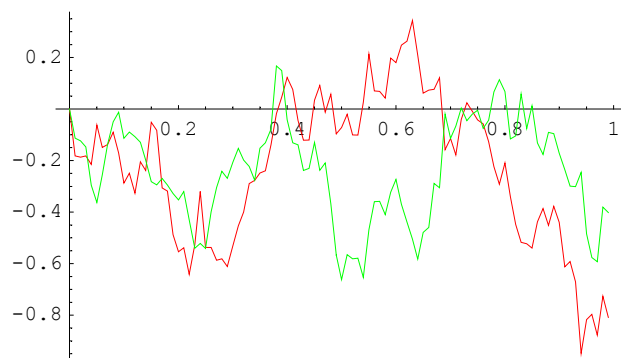
Standardno dvodimenzionalno Brownovo gibanje je dvodimenzionalni stohastički proces $(B_{t,1}, B_{t,2})$ gdje su $B_{t,1}$ i $B_{t,2}$ nezavisna, standardna (jednodimenzionalna) Brownova gibanja.

■ Simulacija dvodimenzionalnog Brownovog gibanja



Trajektorije (nezavisnih) komponenti.





Korelirano dvodimenzionalno Brownovo gibanje

Korelirano dvodimenzionalno Brownovo gibanje je dvodimenzionalni stohastički proces $W=A B + \mu t$, gdje je B standardno dvodimenzionalno Brownovo gibanje, A matrica reda dva, a μ je dvodimenzionalni vektor.

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix};$$

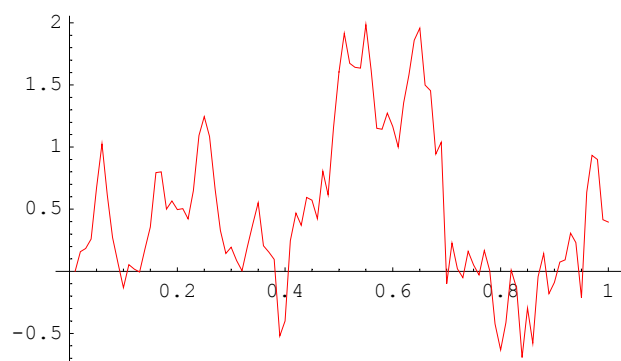
$$\mu = \{0, 0\};$$

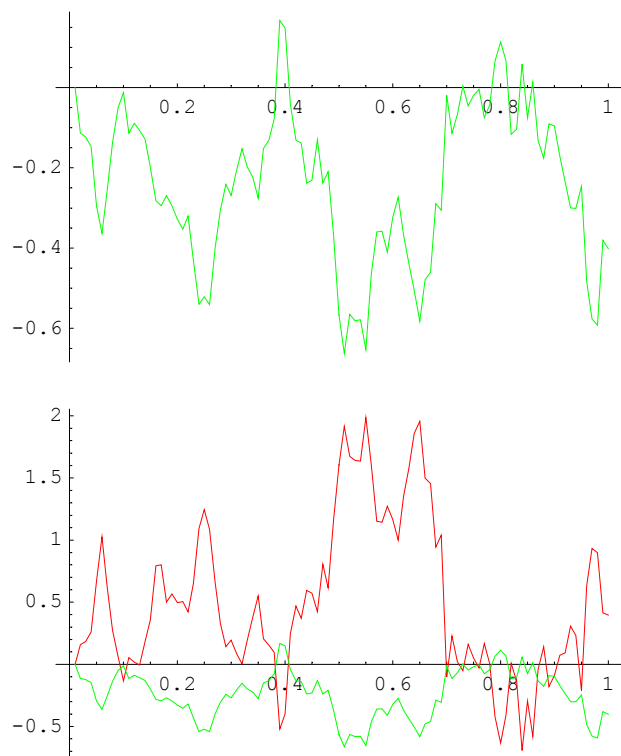
Kovarijacijska matrica jednaka je $\Gamma=A^*A^T$.

$$\Gamma = \text{MatrixForm}[\mathbf{A}.\text{Transpose}[\mathbf{A}]]$$

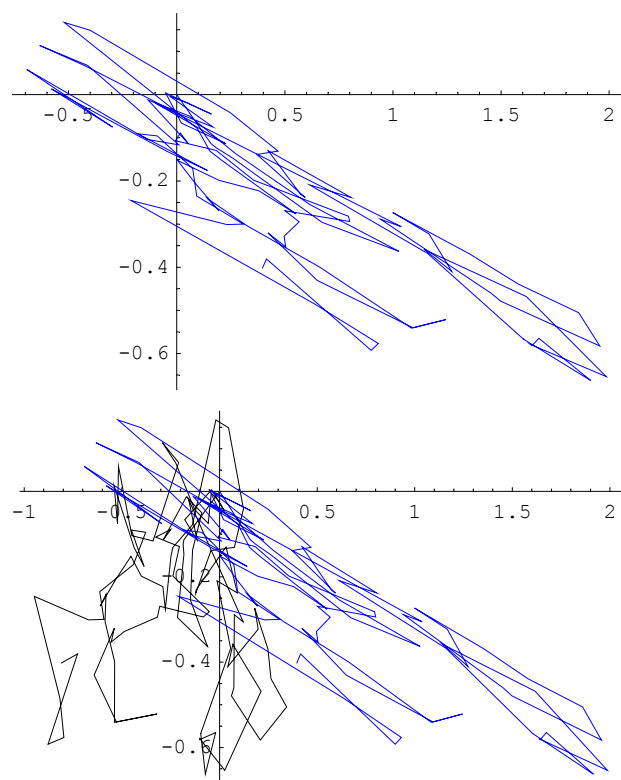
$$\begin{pmatrix} 10 & -3 \\ -3 & 1 \end{pmatrix}$$

Sljedeća dva grafa prikazuju simulirane komponente vektora W , dok su na trećem dane obje komponente.





Sljedeće slike prikazuju korelirano Brownovo gibanje W u ravnini, te zajedno korelirano i nekorelirano Brownovo gibanje u ravnini.



Difuzije

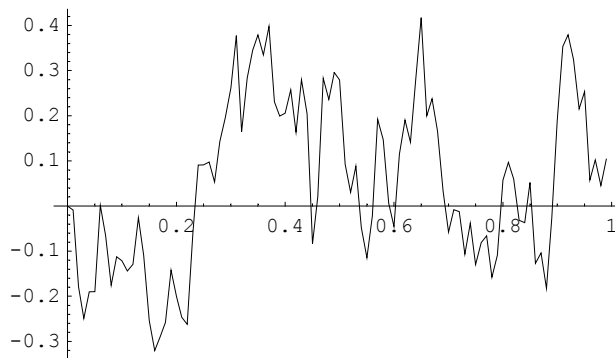
Difuzija $(X_t, t > 0)$ je Markovljev proces s neprekidnim putovima koji zadovoljava sljedeću stohastičku diferencijalnu jednažbu:

$$X_t = X_0 + \int_0^t \mu(X_s, s) ds + \int_0^t \sigma(X_s, s) dB_s$$

gdje je $(B_t, t > 0)$ standardno Brownovo gibanje, a μ i $\sigma > 0$ realne funkcije.

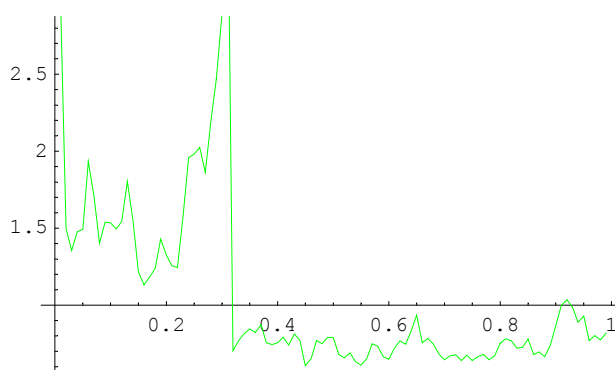
■ Simulacija difuzije

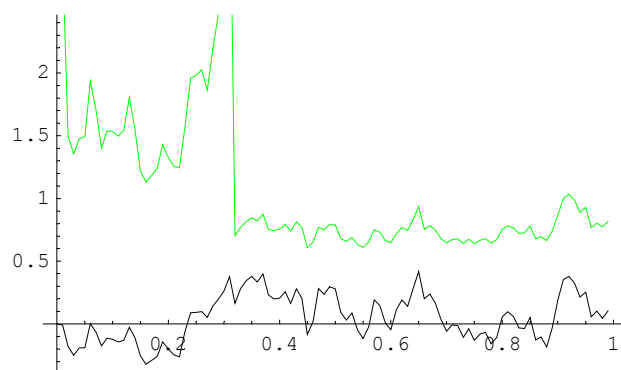
Generirajuće Brownovo gibanje.



Difuzija s $\mu(x)=x$, $\sigma(x)=x^2$ i usporedba s generirajućim Brownovim gibanjem.

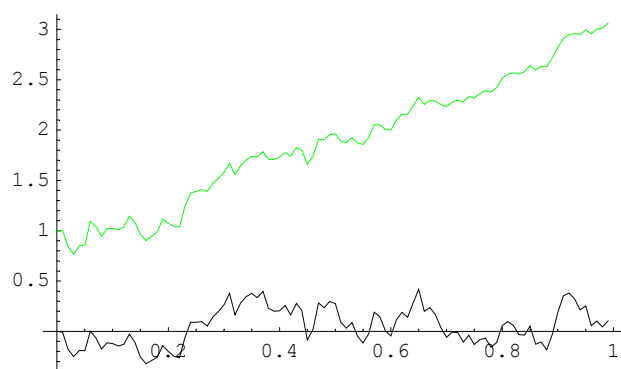
$$dX_t = X_t dt + (X_t)^2 dB_t \quad ; \quad X_0 = 3$$





Difuzija s $\mu(x)=x$, $\sigma(x)=1/x$ i usporedba s generirajućim Brownovim gibanjem.

$$dX_t = X_t dt + \frac{1}{X_t} dB_t \quad ; \quad X_0 = 1$$

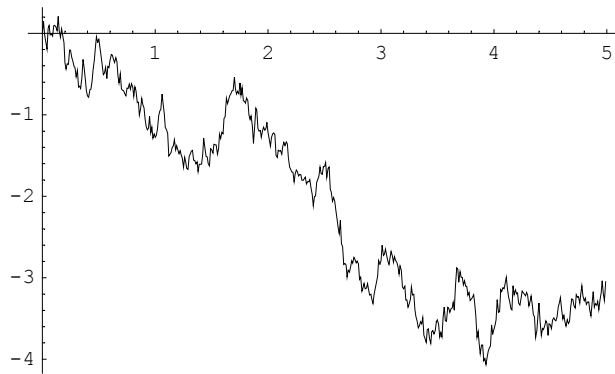


Ornstein-Uhlenbeckov proces

Ornstein-Uhlenbeckov proces je difuzija kod koje je $\mu(x) = -\gamma x$, $\gamma > 0$, A σ je konstanta.

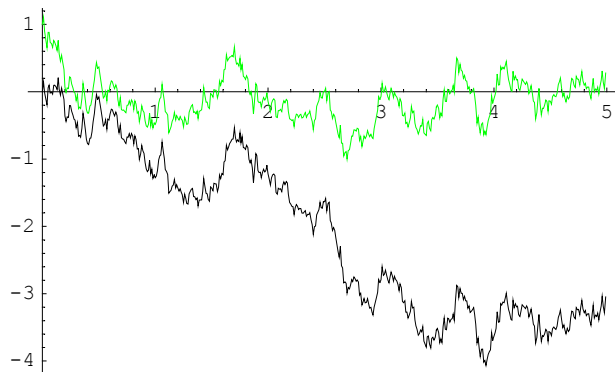
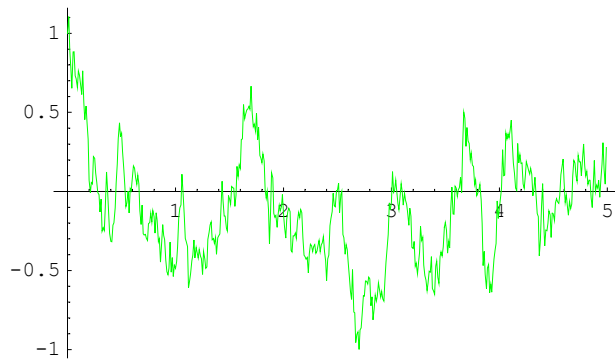
■ Simulacija Ornstein-Uhlenbeckovog procesa

Generirajuće Brownovo gibanje.



Ornstein-Uhlenbeckov proces s $\gamma=4$ i $\sigma=1$, i usporedba s generirajućim Brownovim gibanjem.

$$dX_t = -4 (X_t) dt + 1 dB_t \quad ; \quad X_0 = 1$$



Geometrijsko Brownovo gibanje

Geometrijsko Brownovo gibanje je difuzija X_t kod koje je koeficijent drifta jednak

$\mu(x)=\mu x$, a koeficijent difuzije $\sigma(x)=\sigma x$. Proces X_t zadovoljava sljedeću stohastičku diferencijalnu jednačinu:

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dB_s$$

Rješenje jednačine je:

$$X_t = X_0 \exp \{ (\mu - \sigma^2 / 2) t + \sigma B_t \}$$

Geometrijsko Brownovo gibanje je model kretanja cijena dionica. Uveo ga je Samuelson, a koristili Black, Scholes i Morton.

■ Ponašanje geometrijskog Brownovog gibanja za $t \rightarrow \infty$

Ako je $\mu - \sigma^2 / 2 < 0$, tada $X_t \rightarrow 0$ za $t \rightarrow \infty$.

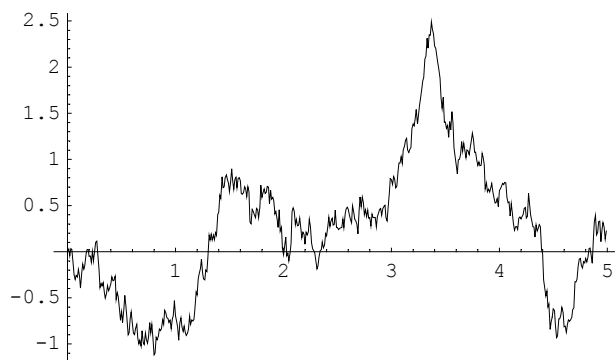
Ako je $\mu - \sigma^2 / 2 = 0$, tada X_t oscilira između 0 i $+\infty$.

Ako je $\mu - \sigma^2 / 2 > 0$, tada $X_t \rightarrow +\infty$ za $t \rightarrow \infty$.

Za $\mu = 0$, X_t je martingal.

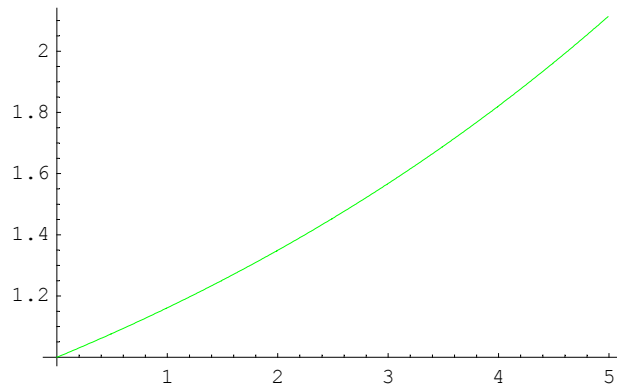
■ Simulacija geometrijskog Brownovog gibanja

Generirajuće Brownovo gibanje.



Geometrijsko Brownovo gibanje sa $\mu=0.15$, te redom $\sigma=0$, $\sigma=0.05$, $\sigma=0.1$, $\sigma=0.2$, $\sigma=0.5$.

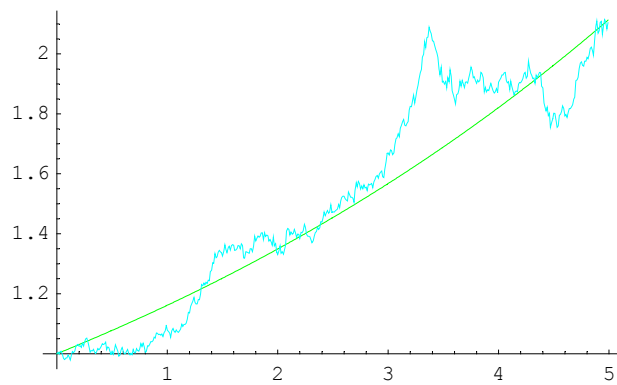
$$dX_t = 0.15 (X_t) dt + \sigma (X_t) dB_t \quad ; \quad X_0 = 1$$



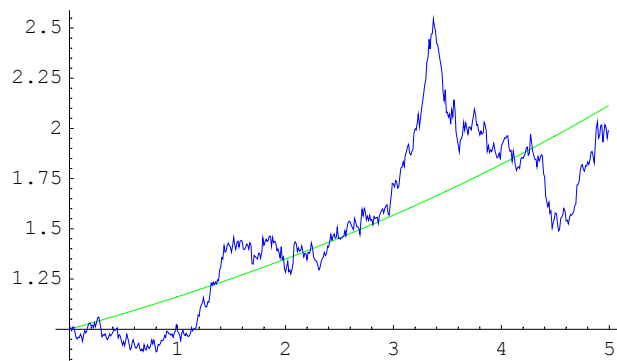
$$dX_t = 0.15 (X_t) dt + 0.05 (X_t) dB_t ; X_0 = 1$$



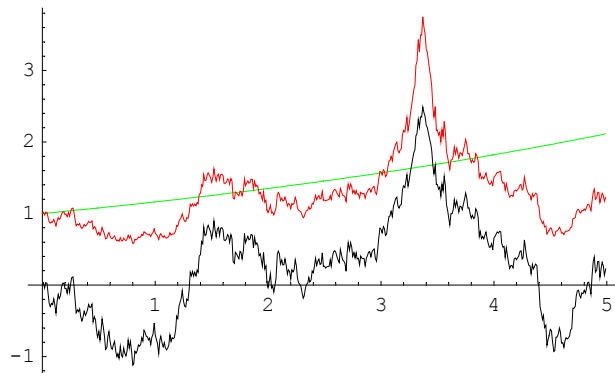
$$dX_t = 0.15 (X_t) dt + 0.1 (X_t) dB_t ; X_0 = 1$$



$$dX_t = 0.15 (X_t) dt + 0.2 (X_t) dB_t ; X_0 = 1$$



$$dX_t = 0.15 (X_t) dt + 0.5 (X_t) dB_t ; X_0 = 1$$



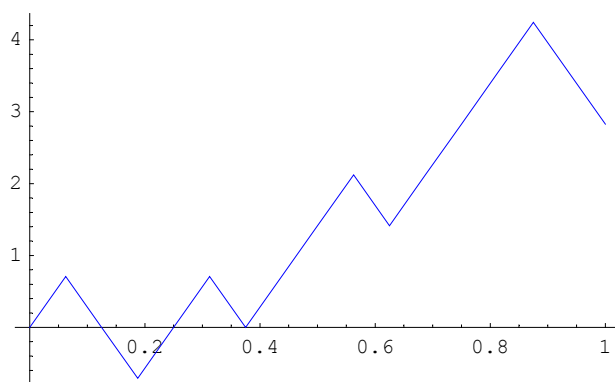
Brownovo gibanje i slučajne šetnje

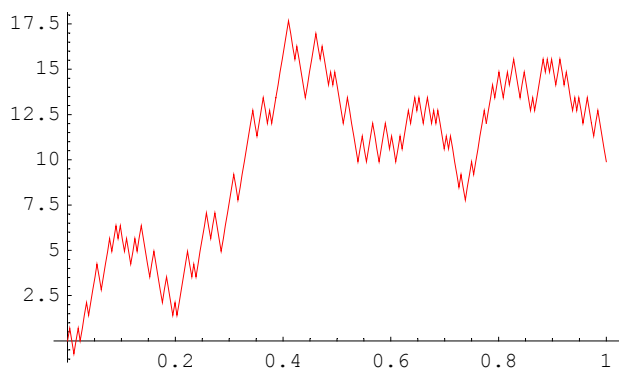
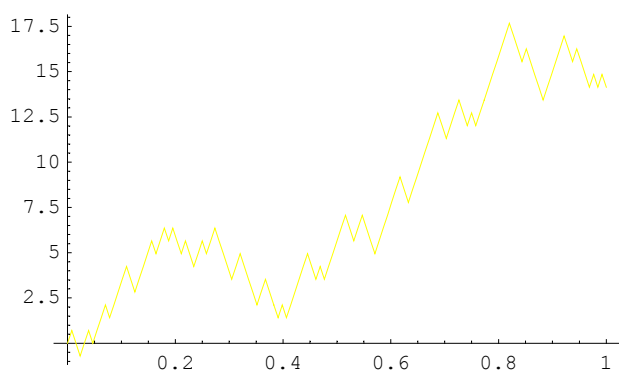
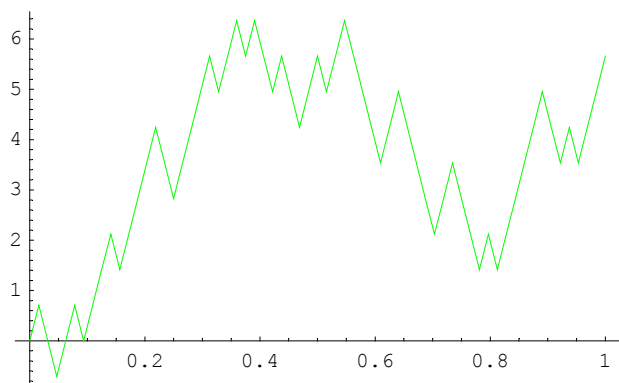
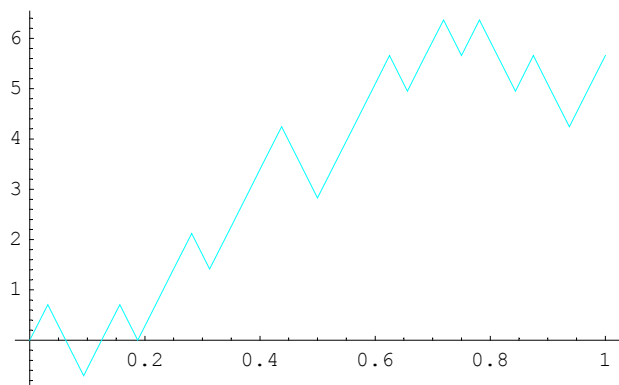
Brownovo gibanje aproksimiramo nizom slučajnih šetnji. Koraci slučajne šetnje su centrirani, s konačnom varijancom. U svim primjerima zahtijevat ćemo da je varijanca jednaka 1.

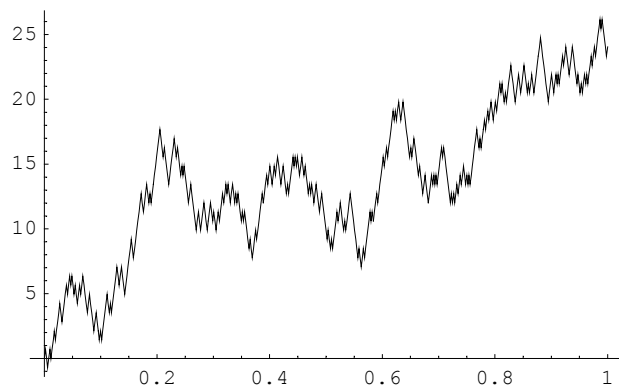
■ Simulacija slučajne šetnje

- Bernoullijevi koraci: $X = \begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ 1/2 & 1/2 \end{pmatrix}$

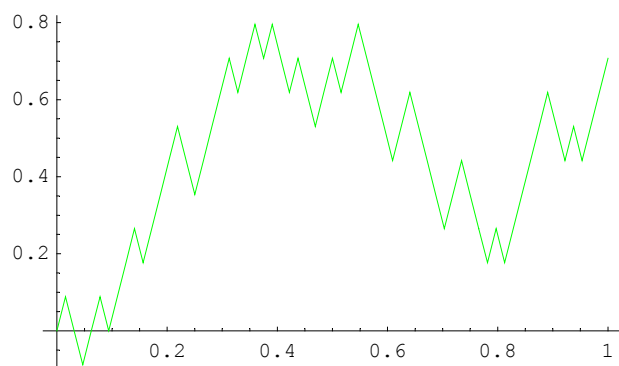
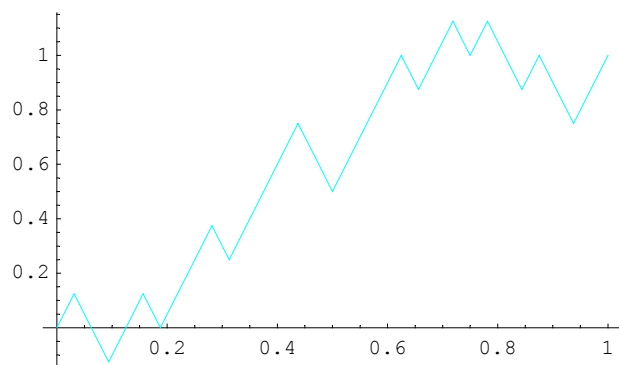
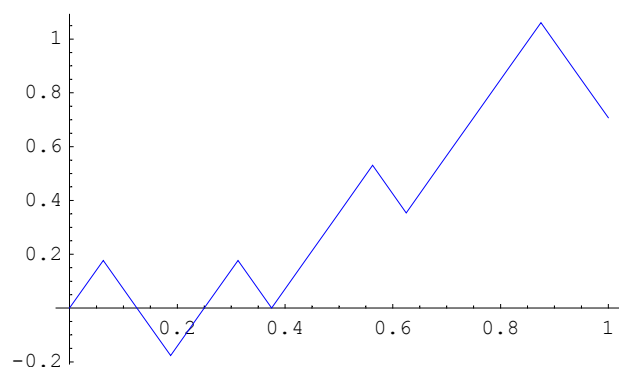
Prostorno neskalirane aproksimacije.

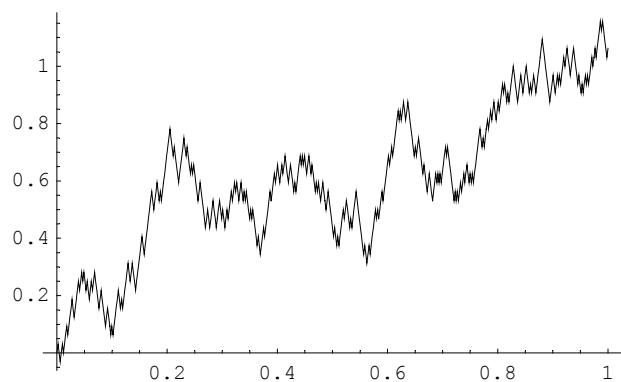
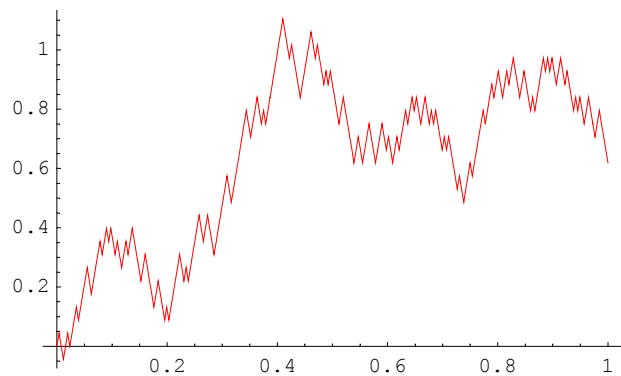
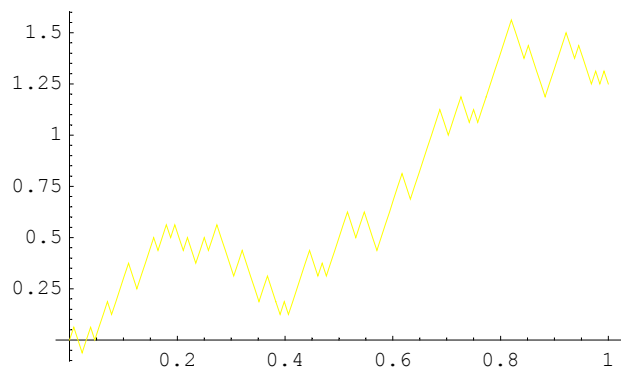




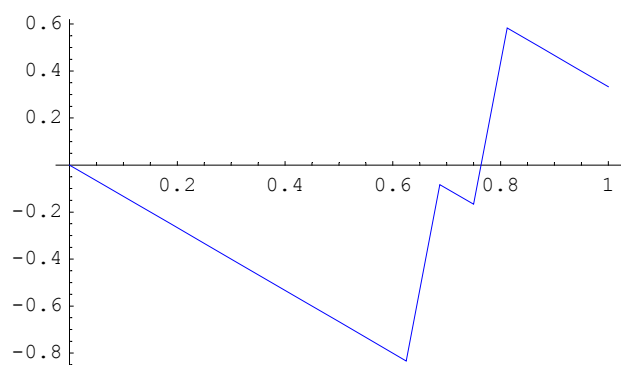


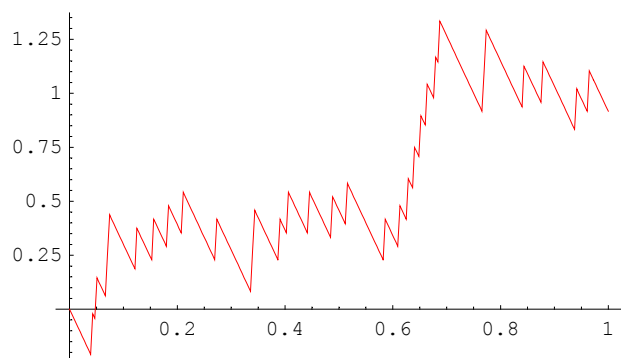
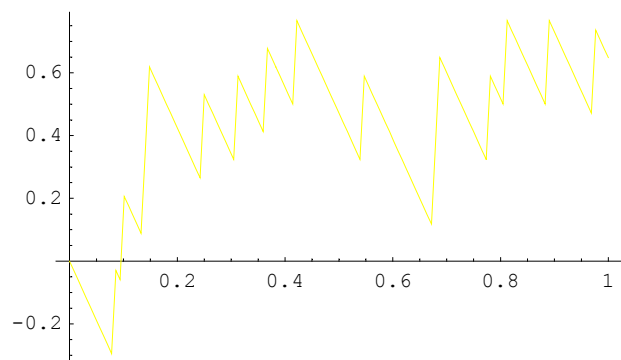
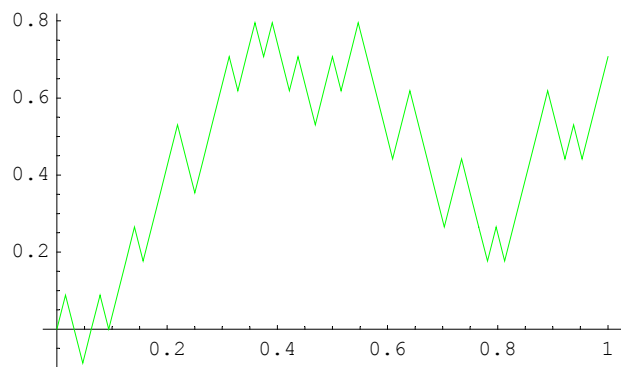
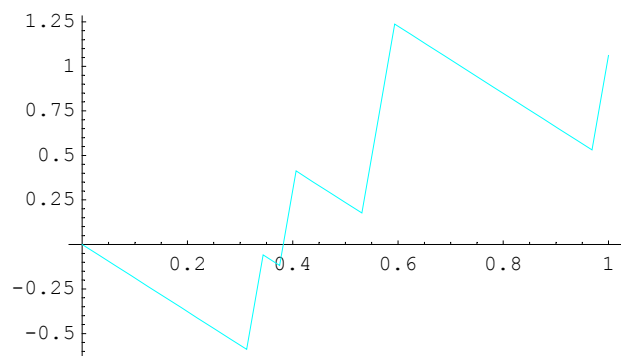
Prostorno skalirane aproksimacije.

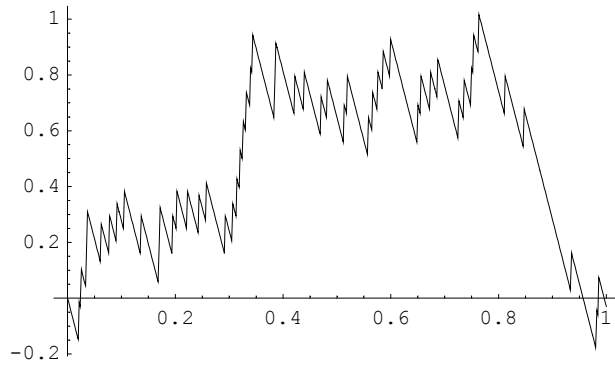




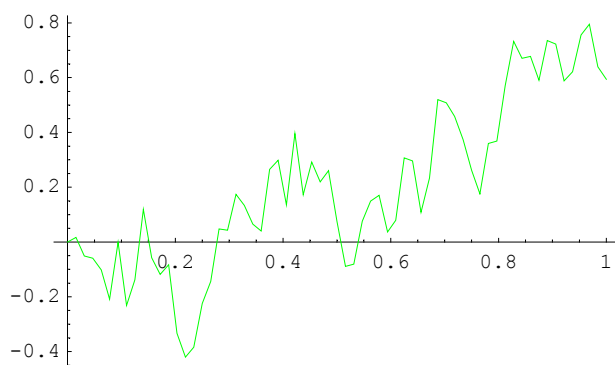
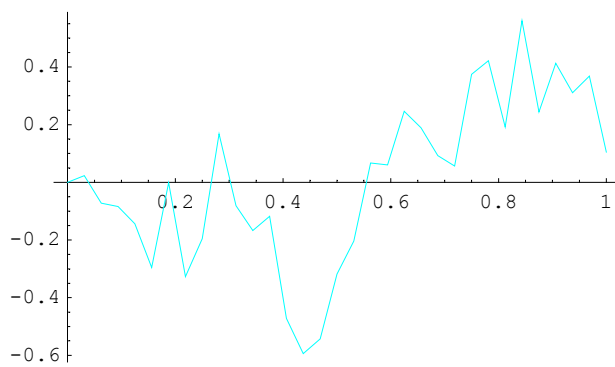
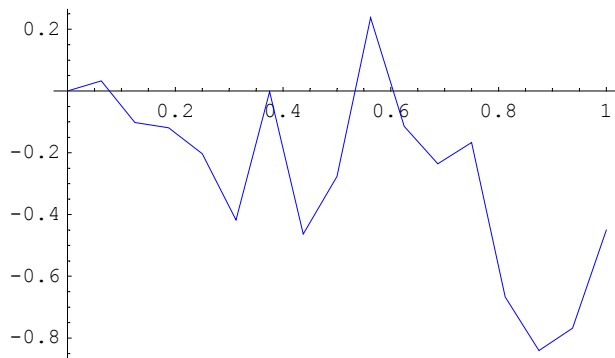
■ Bernoullijevi koraci: $X = \begin{pmatrix} -1/3 & 3 \\ 9/10 & 1/10 \end{pmatrix}$

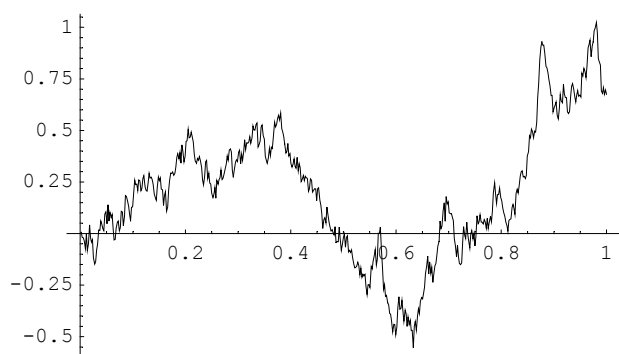
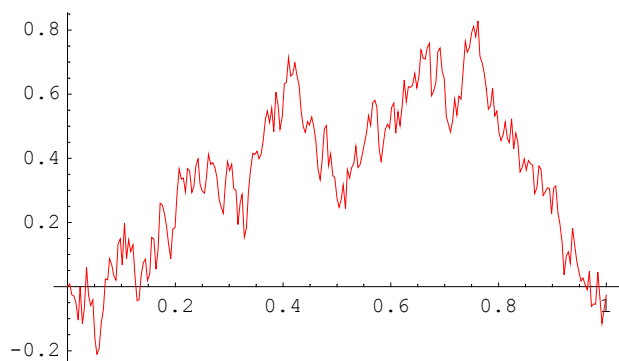
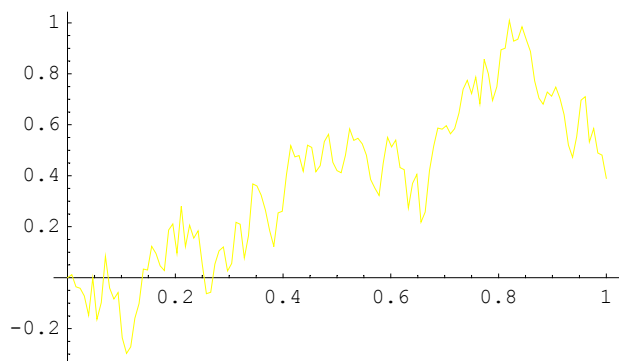






■ Normalni koraci: $X=N(0,1)$





■ Eksponencijalni koraci: $X=E(1)-1$

